A COLLEGE ALGEBRA

BY

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PREFACE

This work, as the name implies, is intended for colleges and scientific schools. The first part is simply a review of the principles of Algebra preceding Quadratic Equations, with just enough examples to illustrate and enforce these principles. By this brief treatment of the first chapters sufficient space is allowed, without making the book cumbersome, for a full discussion of Quadratic Equations, The Binomial Theorem, Choice, Chance, Series, Determinants, and The General Properties of Equations. Every effort has been made to present in the clearest light each subject discussed, and to give in matter and methods the best training in algebraic analysis at present attainable. Many problems and sections can be omitted at the discretion of the instructor.

The author is under great obligation to J. C. Glashan, LL.D., Ottawa, Canada, to Professor J. J. Hardy, Ph.D., Lafayette College, Easton, Pa., and to W. H. Butts, A.M., Michigan University, Ann Arbor, Mich., who have read the proofs and given valuable suggestions on the subject-matter.

Answers to the problems are bound separately in paper covers, and will be furnished free to pupils when *teachers* apply to the *publishers* for them.

Any corrections or suggestions relating to the work will be thankfully received.

G. A. WENTWORTH.

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COLLEGE ALGEBRA

CHAPTER I

FUNDAMENTAL IDEAS

1. Magnitude, Quantity, and Number. Whatever admits of increase or decrease is called a **magnitude**. Every magnitude must therefore admit of comparison with another magnitude of the same kind in such a way as to determine whether the first is greater than, less than, or equal to the other.

A measurable magnitude is a magnitude that admits of being considered as made up of parts all equal to one another.

To **measure** any given measurable magnitude, we take as standard of reference a definite magnitude of the same kind as the magnitude to be measured and determine how many magnitudes, each equal to the standard of reference, will together constitute the given magnitude.

A **quantity** is a measurable magnitude expressed as a *magnitude actually measured.* Hence, the expression of a quantity consists of two components. One of these components is the *name* of the magnitude that has been selected as the standard of reference or measurement. The other component expresses how many magnitudes, each equal to the standard of reference, must be taken to make up the quantity. The standard magnitude is termed a **unit**, and the other component of the expression is termed the **numerical value** of the expression. Hence,

A **unit** is the standard magnitude employed in counting any collection of objects or in measuring any magnitude.

A **number** is that which is applied to a unit to express how many parts, each equal to the unit, there are in the magnitude measured.

The endless succession of numbers *one, two, three, four,* etc., employed in counting is called the **natural series of numbers.**

2. In the statement *James walked 12 miles,* the number of miles is actually stated, and the 12 is therefore called **a known number,** or it is said to be *explicitly assigned.*

In the statement *If from five times the number of miles James walked, ten is subtracted, the remainder will be fifty,* the number of miles, though not *directly* given, may be found from the *data* to be twelve and is therefore said to be implied in the statement, or it is called an *implicitly assigned* number, or more commonly, an **unknown number.**

In the statement *If from the double of a number six is subtracted, the result will be the same as if three had been subtracted from that number and the remainder doubled,* the number to be doubled is assigned neither *explicitly* nor *implicitly,* since the statement is true for any number whatever. A number of this kind, which may have any value whatever, is called an *arbitrary number.* Arbitrary numbers are frequently called **known numbers,** as they are often assumed to be known, though not definitely assigned.

3. Numbers explicitly assigned are represented in Algebra, as they are in Arithmetic, by the numerals or figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and combinations of these. Each figure or combination of figures represents one and but one particular number. Numbers implicitly assigned and arbitrary numbers are usually represented by the letters of the alphabet. The first letters of the alphabet, as a, b, c, etc., are generally used to represent arbitrary numbers, while z, y, x, w, etc., commonly represent unknown numbers.

4. When any letter, as x, is used in the course of a calculation it denotes the same number throughout. We may also

represent different numbers by the same letter with marks affixed.

Thus, instead of writing a, b, c for three different numbers, we may represent these numbers by the symbols a_1, a_2, a_3 (read *a sub-one*, *a sub-two*, etc.), or by a', a'', a''' (read *a prime*, *a second*, etc.).

5. In Arithmetic the figures that represent numbers are generally themselves called numbers; and, similarly, in Algebra the symbols that stand for numbers are themselves called numbers. Letter-symbols are called *literal expressions*, and figure-symbols *numerical expressions*.

The number which a letter represents is called its *value*, and if represented *arithmetically*, its *numerical* value.

6. In elementary Algebra we consider all quantities as expressed numerically in terms of some unit, and the symbols represent only the *purely numerical parts* of such quantities. In other words, the symbols denote what are called in Arithmetic *abstract* numbers.

7. An **algebraic expression** is the expression of a number in algebraic symbols.

8. Certain words and phrases occur so often in Algebra that it is found convenient to represent them by easily made symbols.

Symbols of Relation.

$=$, read *equals, is equal to, will be equal to*, etc.

$\neq$, read *is not equal to*, etc.

$>$, read *is greater than*, thus $9 > 4$.

$<$, read *is less than*, thus $4 < 9$.

$:$, $::$, the signs of proportion, as in Arithmetic.

Thus, $a : b :: c : d$, or $a : b = c : d$, is read a *is to* b *as* c *is to* d.

Symbols for Words.

$\therefore$, read *therefore, consequently, hence.*

$\because$, read *because, since.*

Thus, $\because a = b$, and $b = c$; $\therefore a = c$, is read *since* a *equals* b, *and* b *equals* c; *therefore* a *equals* c.

$\cdots$, the symbol of continuation, is read *continued by the same law.*

Thus, 1, 2, 3, 4, $\cdots$ means that we are to continue the numbers by the same law; $x_1, x_2, x_3, \cdots, x_n$ means x_1, x_2, x_3, x_4, x_5, and so on to x_n.

9. Signs of Operation. The principal operations of Algebra are Addition, Subtraction, Multiplication, Division, Involution, Evolution, and Logarithmation. A mark used to denote that one of these operations is to be performed on a number is called a *sign of operation.* These *signs of operation* will now be explained.

10. The **sign of addition** is $+$ (read *plus*). As in Arithmetic, it denotes that the number before which it stands is an *addend.*

Thus, $a + b$ means that b is to be added to a; so that if a represents 6 and b represents 4, $a + b$ represents $6 + 4$, which is 10. $a + b + c$ denotes that b is to be added to a, and then c added to their sum.

The sum of two or more numbers is expressed by writing them in a row with the sign $+$ before each of them except the first number.

11. The **sign of subtraction** is $-$ (read *minus*). As in Arithmetic, placed before a number it denotes that that number is a *subtrahend.*

Thus, $a - b$ (read *a minus b*), indicates that the number represented by b is to be subtracted from the number represented by a; so that, if a represents 6 and b represents 4, $a - b$ is equivalent to $6 - 4$, which is 2.

Hence, to indicate that a number is to be subtracted from another number, as a from x, *write the subtrahend after the minuend with the sign $-$ between them.*

The expression $a + b - c$ denotes that b is to be added to a, and then c subtracted from the sum. $a - b - c$ denotes that b is to be subtracted from a, and then c subtracted from the remainder.

12. Numbers to be multiplied together are called **factors**, and the resulting number is called the *product* of these factors. Multiplication is indicated in two ways:

1. *By a sign of operation.* 2. *By position.*

The **signs of multiplication** are $\times$ and $\cdot$ (read *into, times*, or *multiplied by*).

Thus, $3 \cdot 4 \cdot 5$, or $3 \times 4 \times 5$ indicates the *continued* product of the three factors 3, 4, and 5. In like manner, $a \cdot b$, or $a \times b$, indicates the product of the factors a and b.

If all the factors or all but one are represented by letters, the *signs of operation*, $\times$ and $\cdot$, are generally omitted; this method is called **indicating multiplication by position.**

Thus, five times a is written $5a$ (read *five a*), and $\frac{3}{8}$ of the product of m and z is written $\frac{3}{8}mz$.

A number which multiplies another number is called a **coefficient** of that number. A *coefficient* (literally, *co-factor*) is therefore simply a multiplier, *numerical* or *literal.*

Thus, in the expression $5amx$,

5	is the	numerical	coefficient	of	amx,
$5a$	"	literal	"	"	mx,
$5am$	"	"	"	"	x.

If no numerical coefficient is written, unity is understood as the actual numerical coefficient.

13. The **sign of division** is $\div$ (read *divided by*), and denotes that the number immediately following it is a divisor.

Thus, $a \div b$ (read *a divided by b*) means that a is to be divided by b. If a represents 12 and b represents 4, $a \div b$ represents $12 \div 4$, or 3.

Division is also indicated by arranging the numbers in the form of a fraction with the *dividend* for *numerator* and the *divisor* for *denominator.*

Thus, $a \div b$ may be written $\frac{a}{b}$; $ax \div by$ may be written $\frac{ax}{by}$.

This method is called **indicating division by position.**

14. In an expression such as $7ax + 5cy - 3dz$ (read *seven* ax *plus five* cy *minus three* dz) *the multiplications are to be performed before the additions and subtractions.*

In an expression such as $\frac{ax}{m} + \frac{by}{n} - \frac{cz}{q}$ *the multiplications and divisions are to be performed before the additions and subtractions,* so that in this expression the quotient of ax by m is to be increased by the quotient of by by n, and the sum diminished by the quotient of cz by q.

15. A **power** of a number is the product obtained by using that number a certain number of times as a multiplier, starting with *unity as first multiplicand.* The operation of forming a power is called **involution**; the number used as a multiplier is called the **base** of the power; the *number of successive multiplications by the base* is called the **degree** of the power; and the number indicating the degree of the power is called the **exponent** or **index** of the power and is written in small characters to the right and a little above the line of the base.

Thus, $1 \times a \times a$ is represented by a^2 (read *a square*); here a is the *base*, 2 is the *exponent* (or *index*), and a^2 is the second *power* of a.

$1 \cdot c \cdot c \cdot c$ is represented by c^3 (read *c cube*); here c is the base, 3 is the exponent, and the number c^3 is the third power of c.

In x^5 (read *x to the fifth*), x is the base, 5 is the exponent, and the number x^5 is the fifth power of x.

Since the exponent denotes how many multiplications by the base are to be made, the first to be performed on unity, it follows that a^1, the first power of a, represents $1 \times a$, or simply a.

Hence, also, a^0, the zero power of a, denotes that *no* multiplication by a is to be made, or, in other words, that the unit-multiplicand is not to be multiplied by a. Therefore $a^0 = 1$ for any value of a whatsoever.

16. In writing a power at full length as a product it is usual to omit the unit-multiplicand, just as it is usual to omit a unit-coefficient where such occurs.

Thus, instead of writing $x^3 = 1 \times x \times x \times x$, we write $x^3 = x \times x \times x$.

In this method of expressing the value of a power *the exponent denotes the number of times the base is taken as a factor.*

17. Comparing powers, the second power is said to be *higher* than the first, the third higher than the second, etc.

18. In an expression such as $4\,a^2b^3 \div c^2$ (read 4 a *square* b *cube divided by* c *square*) *the involutions are to be performed before the multiplications and divisions.*

19. Involution is the operation of forming a power by taking the same number several times as a factor.

Evolution is the inverse of Involution, or the operation of finding one of the *equal factors* of a number. A **root** is one of the equal factors. If the number is resolved into *two equal factors*, each factor is called the **square root**; if into three equal factors, each factor is called the **cube root**; if into four equal factors, each factor is called the **fourth root**; and so on.

The root sign is $\sqrt{\ }$. Except for the square root, a number-symbol is written over the root sign to show into how many equal factors the given number is to be resolved. This number-symbol is called the **index of the root.**

Thus, $\sqrt{64}$ means the square root of 64; $\sqrt[3]{64}$ means the cube root of 64.

20. Logarithmation is the operation of determining the index or exponent which the given base must have in order that the resulting root or power may be equal to a given number. The index or exponent is called the **logarithm** of the given number to the given base.

Thus, if a and b are given numbers and $a^n = b$, n is called the logarithm of b to the base a.

21. Positive and Negative Numbers. There are quantities which stand to each other in such an opposite relation that, when combined, they cancel each other entirely or in part.

Thus, six dollars *gain* and six dollars *loss* just cancel each other; but ten dollars *gain* and six dollars *loss* cancel each other only in part. For the six dollars *loss* will cancel six dollars of the *gain* and leave four dollars gain.

An opposition of this kind exists in *assets* and *debts*, in motion *forwards* and motion *backwards*, in motion *to the right* and motion *to the left*, in the rise *above* zero and the fall *below* zero of the mercury of a thermometer.

From this relation of quantities a question often arises which is not considered in Arithmetic; namely, the subtracting of a greater number from a smaller. This cannot be done in Arithmetic, for the real nature of subtraction consists in *counting backwards* along the natural series of numbers. If we wish to subtract 4 from 6, we start at 6 in the natural series, count four units backwards, and arrive at two, the difference sought. If we subtract 6 from 6, we start at 6 in the natural series, count six units backwards, and arrive at zero. If we try to subtract nine from six, we cannot do it, because, when we have counted backwards as far as zero, *the natural series of numbers has come to an end.*

22. In order to subtract a greater number from a smaller, it is necessary to *assume* a new series of numbers, beginning at zero and extending backwards. If the natural series advances from zero to the right, by repetitions of the unit, the new series must recede from zero to the left, by repetitions of the unit; and the *opposition* between the right-hand series and the left-hand series must be clearly marked. This opposition is indicated by calling every number in the right-hand series a *positive* number, and prefixing to it, when written, the sign +; and by calling every number in the left-hand series a *negative* number, and prefixing to it the sign −. The two series of numbers will be written thus:

$$\cdots -4,\ -3,\ -2,\ -1,\ 0,\ +1,\ +2,\ +3,\ +4, \cdots$$

and may be considered as forming but a single series consisting of a positive portion or branch, a negative portion or branch, and zero. The complete series thus formed is called the **scalar series.**

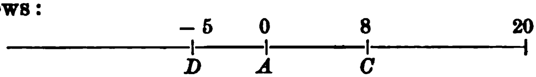
If, now, we wish to subtract 9 from 6, we begin at 6 in the positive branch, count nine units in the *negative direction* (to the left), and arrive at -3 in the negative branch. That is, $6 - 9 = -3$.

The result obtained by subtracting a greater number from a less, when both are positive, is *always a negative number.*

If a and b represent any two numbers of the positive branch, the expression $a - b$ will denote a positive number when a is greater than b; will be equal to zero when a is equal to b; will denote a negative number when a is less than b.

If we wish to add 9 to -6, we begin at -6 in the negative series, count nine units in the *positive direction* (to the right), and arrive at $+3$ in the positive branch.

We may illustrate the use of positive and negative numbers as follows:



Suppose a person starting at A walks 20 feet to the right of A, and then returns 12 feet, where will he be? Answer: at C, a point 8 feet to the right of A. That is, 20 feet − 12 feet = 8 feet; or, $20 - 12 = 8$.

Again, suppose he walks from A to the right 20 feet, and then returns 25 feet, where will he now be? Answer: at D, a point 5 feet to the left of A. That is, if we consider distance measured in feet to the left of A as forming a negative series of numbers, beginning at A, $20 - 25 = -5$. Hence, the phrase, 5 feet to the left of A, is now expressed by the negative number -5.

23. Numbers with the sign + or − are called **scalar numbers.** They are unknown in elementary Arithmetic, but play a very important part in Algebra. Numbers regarded without reference to the signs + or − are called **absolute numbers.**

Every algebraic number, as $+4$ or -4, consists of a sign + or − and the absolute value of the number; in this case 4. The sign shows whether the number belongs to the positive or the negative series of numbers; the absolute value shows

what place the number has in the positive or the negative series.

When no sign stands before a number the sign + is always understood.

Thus, 4 means the same as $+ 4$, a means the same as $+ a$.

But *the sign — is never omitted.*

Two numbers which have, one the sign + and the other the sign —, are said to have **unlike signs.**

Two numbers which have the same absolute values, but unlike signs, always cancel each other when combined.

Thus, $+ 4 - 4 = 0$, $+ a - a = 0$.

24. Meaning of the Signs. The use of the signs + and —, to indicate addition and subtraction, must be carefully distinguished from their use to indicate in which series, the positive or the negative, a given number belongs. In the first sense they are signs of *operations* and are common to both Arithmetic and Algebra. In the second sense they are signs of *opposition* and are employed in Algebra alone.

25. When an expression is made up of several parts connected by the signs +, —, each of these parts taken with the sign immediately preceding it (+ being understood if no written sign precedes) is called a **term.**

Thus, $a + b - c + d + e$ consists of the five *terms* $+a$, $+b$, $-c$, $+d$, $+e$.

A term whose sign is + is called a **positive term**; a term whose sign is — is called a **negative term.**

An expression which consists of but one term is called a **monomial** or **simple expression.**

An expression which consists of two or more terms is called a **polynomial** or **compound expression.**

A polynomial of two terms is called a *binomial.* A polynomial of three terms is called a *trinomial.* Polynomials of three or more terms are sometimes called *multinomials.*

26. If two terms differ only in one having the sign $+$ and the other the sign $-$, they are called **complementary terms.**

Thus, $+b$ and $-b$ are complementary terms in the expression $a+b-b$; so $-c$ and $+c$ are complementary terms in $a-c+b+c$.

27. The **degree** of a *term* is the number of *literal* factors it contains, and each literal factor is called a **dimension** of the term.

Thus, $3a^2b^2c^3$ is of *seven* dimensions.

This term, $a^2b^2c^3$, is said also to be of *two* dimensions in a, of *two* dimensions in b, and of *three* dimensions in c.

The dimensions of a *polynomial* are determined by the dimensions of its *highest term.*

Thus, $1+a^2+3abc$ is of three dimensions because its highest term, $3abc$, is of three dimensions.

A polynomial is said to be **homogeneous** when all its terms have the *same* dimensions.

Thus, $x^3+3x^2y+3xy^2+y^3$ is homogeneous.

28. Like terms are terms that have the same letters, and the corresponding letters have the same exponents.

Thus, $5a^2b^3$, $3a^2b^3$, $-7a^2b^3$ are *like* terms; but $3a^2b$ and $3ab^2$ are *unlike* terms because, though they contain the same letters, the corresponding letters do not have the same exponents.

29. If an expression contains any *like* terms, these may be united, and the expression is said to be **simplified.**

Thus, as in Arithmetic, 2 dozen + 3 dozen = 5 dozen; 2 times 8 + 3 times 8 = 5 times 8; so in Algebra, $2ab+3ab=5ab$; $2a^2b^3+3a^2b^3=5a^2b^3$.

Similarly, in the case of *negative* terms; $5ab-3ab=2ab$; $5a^2b^3-3a^2b^3=2a^2b^3$. Hence,

To reduce two or more like terms to a single equivalent term,

Form the sum of the numerical coefficients of the positive terms and also of the negative terms, then take the difference of these sums, affix the literal parts and prefix to the result the sign of those terms whose numerical coefficients give the greater sum.

Thus, in the expression $5a^2b-7ac^2+3a^2b-6ac^2-4a^2b-6a^2b+15ac^2$, the sum of the coefficients of the *positive* terms in a^2b is 8, and the sum of the coefficients of the *negative* terms is 10; the difference of these is 2, to which we affix the literal part a^2b, getting $2a^2b$; and as the sum (10) of the coefficients of the negative terms is the greater, we prefix the sign $-$, getting $-2a^2b$; similarly, combining the terms in ac^2, we get $+2ac^2$, and the whole expression is *simplified* to $-2a^2b+2ac^2$, or $2ac^2-2a^2b$.

30. The **reciprocal** of a number is 1 divided by that number.

Thus, the reciprocal of a is $\frac{1}{a}$; the reciprocal of a^2b^2 is $\frac{1}{a^2b^2}$.

The product of any number and its reciprocal is *unity*.

Thus, $b \times \frac{1}{b} = 1$.

Hence, a *divisor* may be replaced by its fractionally expressed reciprocal as a *multiplier*. If, for example, the product of a and b is to be divided by m, and the quotient divided by n, this may be represented by

$$ab \div m \div n, \text{ or by } ab \times \frac{1}{m} \times \frac{1}{n}, \text{ or by } \frac{ab}{mn}.$$

31. Compound Expressions. Every algebraic expression, however complex, represents a number and may be treated in any operation as a **single symbol**. If an expression is to be so treated, it is generally enclosed in *brackets;* or a line called a *vinculum* is drawn over it.

Thus, $7+(8-3)$ denotes that 3 is to be subtracted from 8 and the remainder added to 7.

$7-(8-3)$ denotes that 3 is to be subtracted from 8 and the remainder subtracted from 7.

$7 \cdot (8-3)$ or $7 \cdot \overline{8-3}$ means that 3 is to be subtracted from 8 and the remainder multiplied by 7.

Similarly, suppose $a+b-c$ is to be operated on as a single symbol; then,

$x+(a+b-c)$ denotes that the number is to be added to x,
$x-(a+b-c)$ " " " " subtracted from x,
$x(a+b-c)$ " " " " multiplied by x,

$(a+b-c) \div x$	denotes that the number is to be	divided by x,
$(a+b-c)^3$	" " " "	cubed,
$\sqrt[3]{(a+b-c)}$	" "	cube root of the number is to be extracted.

32. An expression that has *a part enclosed in brackets* may itself be enclosed in brackets to form part of a *longer expression;* this again may be enclosed in brackets to form part of a still longer expression; and so on to any extent. When several pairs of brackets are thus employed it is usual to make each pair different from the others in size or shape.

1. If it were required to multiply $a+b$ into the sum of $a(x+y)$ and $b(x-y)$, the result would be expressed thus,

$$(a+b)\{a(x+y)+b(x-y)\}.$$

2. $10a-[5b-\{4c+2(3b-a)\}]$ denotes that a is to be subtracted from $3b$, that the remainder is to be doubled, that the product is to be added to $4c$, that the sum is to be subtracted from $5b$, that the remainder is to be subtracted from $10a$.

Hence, if $a=15$, $b=6$, and $c=1$, we have

$$\begin{aligned}10a-[5b-\{4c+2(3b-a)\}]\\ &=150-[30-\{4+2(18-15)\}]\\ &=150-[30-\{4+2\times 3\}]\\ &=150-[30-\{4+6\}]\\ &=150-[30-10]\\ &=150-20\\ &=130.\end{aligned}$$

33. Substitution. Two quantities, two numbers, or two operations are equal if either can be substituted for the other in algebraic expressions without changing the values of the expressions. From this it follows at once that

Numbers that are equal to the same number are equal to one another.

In symbols: If $a=c$ and $b=c$, then $a=b$.

CHAPTER II

THE ELEMENTARY OPERATIONS

34. The introduction of negative numbers requires the meanings of addition, subtraction, multiplication, and division to be made wider and more comprehensive in Algebra than they are in Arithmetic, but these enlarged meanings must be consistent with the older arithmetical meanings, and the elementary operations when thus generalized must still conform to the fundamental laws which govern these operations in Arithmetic. We now proceed to state these fundamental laws and to explain these wider meanings.

ADDITION

35. In Algebra, as in Arithmetic, numbers which are to be added are called *addends*, and the result of the addition is termed *the sum* of the addends; but it must be borne in mind that in Algebra under the term *numbers* are included not only the numbers indicated by single letters but also those which are the arithmetical values of compound algebraic expressions, just as in Arithmetic numbers are expressed either by single digits or by combinations of digits.

Addition is the operation of combining two or more numbers or algebraic expressions into a single number or expression according to the following laws:

I. *If equal numbers are added to equal numbers the sums are equal.*

If the sum of one pair of addends is equal to the sum of a second pair, and either addend in the first pair is equal to the

corresponding addend in the second pair, the remaining addend in the first pair is equal to the remaining addend in the second pair.

II. ***The sum of two addends is the same, whether the second addend is added to the first, or the first addend is added to the second.***

III. ***The sum of three addends is the same, whether the sum of the second and third addends is added to the first, or the third addend is added to the sum of the first and second.***

IV. ***Adding zero to any number leaves the number unchanged.***

36. These laws expressed in algebraic symbols are:

Suppose a, b, c, and d have each one and *only one* value, zero being a possible value for any one or more of them.

I. If $a = c$ and $b = d$, then $a + b = c + d$.

If $a = c$ and $a + b = c + d$, then $b = d$.

If $b = d$ and $a + b = c + d$, then $a = c$.

Hence, **Addition is *completely* uniform.**

II. $$a + b = b + a.$$

Proposition II is expressed by,

Addition is commutative.

III. $$a + (b + c) = (a + b) + c.$$

Proposition III is expressed by,

Addition is associative.

IV. $$a + 0 = a.$$

The modulus of addition is zero.

37. Cor. 1.
$$\begin{aligned}(a + c) + b &= a + (c + b) && \text{(III)}\\ &= a + (b + c) && \text{(II)}\\ &= (a + b) + c. && \text{(III)}\end{aligned}$$

Hence, *adding any number to an addend adds an equal number to the sum.*

Cor. 2. If $a + b = a$, then $b = 0$. (IV and I)

Zero is the only addend whose addition to a number leaves the number unchanged.

38. An algebraic number which is to be added or subtracted is often enclosed in brackets, in order that the signs + and − which are used to distinguish positive and negative numbers may not be confounded with the + and − signs that denote the operations of addition and subtraction.

Thus, $+4 + (-3)$ expresses the sum of the numbers $+4$ and -3; and $+4 - (-3)$ expresses that -3 is to be subtracted from $+4$.

39. Monomials. In order to add two algebraic numbers, we begin at the place in the scalar series which the first number occupies and count, *in the direction indicated by the sign of the second number*, as many units as there are units in the absolute value of the second number.

$$\cdots -4,\ -3,\ -2,\ -1,\ 0,\ +1,\ +2,\ +3,\ +4, \cdots$$

Thus, the sum of $+4 + (+3)$ is found by counting from $+4$ three units in the *positive* direction and is, therefore, $+7$; the sum of $+4 + (-3)$ is found by counting from $+4$ three units in the *negative* direction and is, therefore, $+1$.

In like manner, the sum of $-4 + (+3)$ is -1, and the sum of $-4 + (-3)$ is -7.

1. To add two numbers with *like* signs, find the *sum* of their absolute values, and prefix the common sign to the sum.

2. To add two numbers with *unlike* signs, find the *difference* between their absolute values, and prefix to the difference the sign of the number that is the greater in absolute value.

Thus, (1) $+a + (+b) = a + b$; (3) $-a + (+b) = -a + b$;

(2) $+a + (-b) = a - b$; (4) $-a + (-b) = -a - b$.

By successive application of the above rules we readily obtain rules for adding any number of terms.

Thus,
$$4a + 5a + 3a + 2a = 14a;$$
$$-3a - 15a - 7a + 14a - 2a = 14a - 27a = -13a;$$
$$4a - 3b - 9a + 7b = -5a + 4b.$$

40. Polynomials. Two or more polynomials are added by adding their separate terms.

It is convenient to arrange the terms in columns, so that like terms shall stand in the same column.

Thus,
$$\begin{array}{r}
2a^3 - 3a^2b + 4ab^2 + b^3 \\
a^3 + 4a^2b - 7ab^2 - 2b^3 \\
-3a^3 + a^2b - 3ab^2 - 4b^3 \\
2a^3 + 2a^2b + 6ab^2 - 3b^3 \\
\hline
2a^3 + 4a^2b - 8b^3
\end{array}$$

Addition in Algebra does not necessarily imply *augmentation*, as it does in Arithmetic.

Thus, $7 + (-5) = 2$.

The word **sum**, however, is used to denote the result.

Such a result is called the **algebraic sum**, when it is necessary to distinguish it from the *arithmetical sum*, which would be obtained by adding the *absolute values* of the numbers.

Exercise 1

Add:

1. $9a^2 + 3a + 4b,\ 2a^2 - 4a + 5b,\ 5a - 2b - 6a^2.$
2. $7x^2 - 2xy + y^2,\ 4xy - 2y^2,\ 8x^2 - 9xy + 12y^2.$
3. $7a^2b + 9ab^2 - 13b^3,\ 3a^3 + 2ab^2 - 7b^3,\ ab^2 - a^2b - 6a^3,$
$5b^3 - 7a^3 - ab^2,\ 4b^3 - 2a^3 + a^2b.$
4. $5x^4 + 2x^2 - 7,\ 4x^3 + x - 9,\ 1 + x - x^2,$
$x^5 + x^4 - x^3 - x^2 - 7,\ 9x^2 + 9x^3 - 12x - 4x^4 + 10.$
5. $3m^4 + 2m^3n + 5m^2n^2 - 9n^4,\ 7n^4 - 3mn^3 - 8m^2n^2,$
$11mn^3 - 4m^2n^2 + 6m^3n,\ 5m^4 + 2m^3n - 15mn^3 - 7n^4.$
6. $2x^6 + 3x^5y - 4x^4y^2,\ 2y^6 - 3xy^5 + 4x^2y^4 - 10x^3y^3,$
$5x^3y^3 + 4x^2y^4 - 9y^6,\ 8x^5y - 7x^4y^2 + 6x^3y^3 - 8x^2y^4.$

SUBTRACTION

41. **Subtraction is the operation by which, when the sum of two addends and one of the addends are given, the other addend is determined. In symbols: Subtraction is the operation symbolized by $a - b$, such that**

$$(a-b)+b=a;$$

and by $b-a$, such that

$$a+(b-a)=b.$$

With reference to this operation, the sum is called the **minuend**, the given addend is called the **subtrahend**, and the required addend is called the **remainder**.

42. The laws of subtraction are not fundamental but are derived from this definition combined with the laws of addition. They are:

i. *If equals are subtracted from equals, the remainders are equal.*

ii. *Subtracting any number from an addend subtracts an equal number from the sum.*

iii. *Adding any number to the minuend adds an equal number to the remainder.*

iv. *Subtracting any number from the minuend subtracts an equal number from the remainder.*

v. *Adding any number to the subtrahend subtracts an equal number from the remainder.*

vi. *Subtracting any number from the subtrahend adds an equal number to the remainder.*

43. These laws expressed in algebraic notation are:

Suppose a, b, c, and d have each one and *only one* value, zero included as a possible value for any one or more of them.

If $\quad a = c$ and $b = d$,

$$a - b = c - d. \qquad \text{(i)}$$

$$(a - c) + b = (a + b) - c,$$

and $$a + (b - c) = (a + b) - c. \qquad \text{(ii)}$$

$$(a + c) - b = (a - b) + c. \qquad \text{(iii)}$$

$$(a - c) - b = (a - b) - c. \qquad \text{(iv)}$$

$$a - (b + c) = (a - b) - c. \qquad \text{(v)}$$

$$a - (b - c) = (a - b) + c. \qquad \text{(vi)}$$

44. By definition, $(a - b) + b = a$.

Therefore, if $\quad b = 0$,

$$(a - 0) + 0 = a.$$

That is, $\quad a - 0 = a. \qquad$ (IV, p. 15)

Conversely, if $\quad a - b = a$, then $b = 0$,

for in this case $\quad (a - b) + b = (a - b)$,

and therefore $\quad b = 0. \qquad$ (Cor. 2, p. 16)

45. Monomials. In order to find the difference between two algebraic numbers, we begin *at the place in the scalar series which the minuend occupies* and *count in the direction opposite to that indicated by the sign of the subtrahend* as many units as there are units in the absolute value of the subtrahend.

Thus, when we subtract $+3$ from $+4$ we count from $+4$ three units in the *negative* direction, and arrive at $+1$; when we subtract -3 from $+4$ we count from $+4$ three units in the *positive* direction, and arrive at $+7$. In like manner, $+3$ from -4 is -7; -3 from -4 is -1.

Hence,

1. Subtracting a *positive* number is equivalent to adding an equal *negative* number.

2. Subtracting a *negative* number is equivalent to adding an equal *positive* number.

To subtract one algebraic number from another,

Change the sign of the subtrahend and then add the subtrahend to the minuend.

Thus, (1) $+a-(+b)=a-b$; (3) $-a-(+b)=-a-b$;
(2) $+a-(-b)=a+b$; (4) $-a-(-b)=-a+b$.

46. Polynomials. When one polynomial is to be subtracted from another place its terms under the like terms of the other, change the signs of the subtrahend, and add.

From $\qquad 4x^3-3x^2y-xy^2+2y^3$
take $\qquad 2x^3-x^2y+5xy^2-3y^3$.

Change the signs of the subtrahend and add:

$$\begin{array}{r}4x^3-3x^2y-xy^2+2y^3\\ -2x^3+x^2y-5xy^2+3y^3\\ \hline 2x^3-2x^2y-6xy^2+5y^3\end{array}$$

In practice, instead of actually changing the signs of the subtrahend we only *conceive* them to be changed.

47. Parentheses. Propositions III, p. 15, and ii, v, and vi, p. 19, may be written

$$a+(+b+c)=a+b+c,$$
$$a+(+b-c)=a+b-c,$$
$$a-(+b+c)=a-b-c,$$
$$a-(+b-c)=a-b+c,$$

and, therefore, by § 43, p. 19, and IV, p. 15, and Cor. 2, p. 16,

$$a+(-b+c)=a-b+c,$$
$$a-(-b+c)=a+b-c.$$

Hence, when the parenthesis enclosing a polynomial is preceded by *a plus sign* the parenthesis and plus sign may be removed or omitted without making any change in the signs of the terms of the enclosed polynomial other than inserting the sign + before the first term if that term has no sign expressed.

When a parenthesis enclosing a polynomial is preceded by *a minus sign* the parenthesis and minus sign may be removed *if the signs of the terms of the enclosed polynomial are all changed.*

48. Expressions often occur with more than one parenthesis. These parentheses may be removed in succession by removing *first the innermost parenthesis;* next, the innermost of all that remain, and so on.

Thus,
$$\begin{aligned} a - [b - \{c + (d - \overline{e - f})\}] & \\ &= a - [b - \{c + (d - e + f)\}] \\ &= a - [b - \{c + d - e + f\}] \\ &= a - [b - c - d + e - f] \\ &= a - b + c + d - e + f. \end{aligned}$$

49. The rules for introducing parentheses follow directly from the rules for removing them:

1. Any number of terms of an expression may be put within a parenthesis, and the sign $+$ placed before the whole.

2. Any number of terms of an expression may be put within a parenthesis, and the sign $-$ placed before the whole; *if the sign of every term within the parenthesis is changed.*

Thus,
$$\begin{aligned} a + b - c - d &= (a + b) - (c + d) \\ &= a + (b - c) - d \\ &= a + (b - c - d). \end{aligned}$$

50. By II, p. 15, and ii and iv, p. 19,

$$a + b = b + a,$$
$$a - c + b = a + b - c,$$
$$a - c - b = a - b - c.$$

Hence, **the terms of any polynomial may be combined in any order whatever.**

Thus,
$$\begin{aligned} a + b - c - d &= a - d + (b - c) \\ &= a - c - (d - b) \\ &= -(c - b) - (d - a), \text{ etc.} \end{aligned}$$

Exercise 2

1. From $4a + 5b - 3c$ take $2a + 9b - 8c$.

2. From $7x^3 - x^2 + 4x - 2$ take $2x^3 + 8x^2 - 9x + 8$.

3. From $3a^3 + 3a^2b - 9ab^2 + 3b^3$
take $2a^3 - 5a^2b + 7ab^2 - 9b^3$.

4. From $\frac{1}{3}ab + 4a^2 - \frac{2}{5}b^2 + \frac{1}{4}a$ take $a^2 - \frac{1}{10}b^2 + \frac{1}{8}a$.

5. From $4x^3 - 6x^2 + 8x - 7$ take the sum of
$8x^3 + 7 - 8x^2 + 7x$ and $-9x^3 - 8x^2 + 4x + 4$.

Simplify:

6. $2 - 3x - (4 - 6x) - \{7 - (9 - 2x)\}$.

7. $3a - (a - b - c) - 2\{a + c - 2(b - c)\}$.

8. $4a - [3a - \{2a - (a - b)\} + 5b]$.

9. $[8a - 3\{a - (b - a)\}] - 4[a - 2\{a - 2(a - b)\} + b]$.

10. $x(y + z) + y[x - (y + z)] - z[y - x(z - x)]$.

11. $2x^3(x - 3a) - 2[2x^4 - a^2(x^2 - a^2)]$
$- 3a[x^3 - 2x\{a^2 + x(a - x)\} + a^3]$.

MULTIPLICATION

51. In Algebra, as in Arithmetic, numbers which are to be multiplied together are called *factors*, and the result of the multiplication is termed *the product* of the factors. Under the term *numbers* we include not only the numbers symbolized by single letters but also those which are the arithmetical values of compound algebraic expressions. In the case of two factors, the factor which is to be multiplied by the other is called the *multiplicand*, and the factor by which the multiplicand is to be multiplied is called the *multiplier*.

Multiplication is the operation of combining two or more numbers or algebraic expressions into a single number or expression according to the following laws:

I. *If equal numbers are multiplied by equal numbers, the products are equal.*

If the product of one pair of factors is equal to the product of a second pair, and if either factor in the first pair is equal to the corresponding factor in the second pair and is not zero, the remaining factor in the first pair is equal to the remaining factor in the second pair.

II. *The product of two factors is the same whether the first factor is multiplied by the second or the second factor is multiplied by the first.*

III. *The product of three factors is the same whether the first factor is multiplied by the product of the second and third or the product of the first and second factors is multiplied by the third.*

IV. *Multiplying by unity leaves the multiplicand unchanged.*

V. *If the multiplier is zero, the product is zero.*

52. These laws expressed in algebraic symbols are:

Suppose a, b, c, and d have each one, and *only one*, value, zero included as a possible value for any one or more of them except where noted.

I. If $\quad a = c$ and $b = d$, then $a \times b = c \times d$.

If $\quad a = c \neq 0$, and $a \times b = c \times d$, then $b = d$;

and if $\quad b = d \neq 0$, and $a \times b = c \times d$, then $a = c$.

These propositions are condensed into the single statement,

Multiplication is *completely* uniform for actual or non-zero factors.

II. $$a \times b = b \times a.$$

Proposition II is expressed by,

Multiplication is commutative.

III. $$a \times (b \times c) = (a \times b) \times c.$$

Proposition III is expressed by,

Multiplication is associative.

IV. $$a \times 1 = a.$$

The modulus of multiplication is unity.

V. $$a \times 0 = 0.$$

The annihilator of multiplication is zero.

53. Cor. 1. *Multiplying a factor by any number multiplies the product by that number.*

Cor. 2. *If the product of two factors is equal to one of the factors, the other factor is unity, the case of the product and its equal factor both being zero excepted.*

Cor. 3. *If the product of two or more factors is zero, one at least of the factors is zero.*

Proofs of these corollaries are similar to the proofs in § 37, p. 15.

54. The fundamental law connecting the operation of multiplication with the operations of addition and subtraction is:

VI. *Multiplying the several terms of a polynomial by any number multiplies the polynomial by that number.*

In symbols: $ad + bd - cd = (a + b - c)d.$

Proposition VI is expressed by,

Multiplication is distributive, relative to addition and subtraction.

Hence,
$$\begin{aligned}(a+b)(m+n) &= a(m+n) + b(m+n) && \text{(VI, p. 24)}\\ &= (m+n)a + (m+n)b && \text{(II, p. 23)}\\ &= ma + na + mb + nb && \text{(VI, p. 24)}\\ &= am + an + bm + bn. && \text{(II, p. 23)}\end{aligned}$$

Also, $(a-b)(m-n) = a(m-n) - b(m-n)$ (VI, p. 24)

$= (m-n)a - (m-n)b$ (II, p. 23)

$= ma - na - (mb - nb)$ (VI, p. 24)

$= ma - na - mb + nb$ (vi, p. 19)

$= am - an - bm + bn.$ (II, p. 23)

55. Law of Signs. Let $(+a)$ and $(+c)$ denote positive scalar numbers whose product is $+ac$, and $(-b)$ and $(-d)$ denote negative scalar numbers, a, b, c, and d being the absolute values of the numbers without reference to the relation positive-negative, then

$$(+a)+(-b)=(+a)-(+b),$$

and $$(+c)+(-d)=(+c)-(+d).$$

$$\therefore \{(+a)+(-b)\}\{(+c)+(-d)\}$$
$$=\{(+a)-(+b)\}\{(+c)-(+d)\}. \quad \text{(I, p. 23)}$$

By § 54, $\{(+a)+(-b)\}\{(+c)+(-d)\}$

$$=(+a)(+c)+(+a)(-d)$$
$$+(-b)(+c)+(-b)(-d). \qquad [1]$$

Since $(+a)(+c)=(+ac)$,

$$\{(+a)-(+b)\}\{(+c)-(+d)\}$$
$$=(+ac)-(+ad)-(+bc)+(+bd)$$
$$=(+ac)+(-ad)+(-bc)+(+bd). \qquad [2]$$

Compare the right-hand members of [1] and [2], term by term.

Since $(+a)\times(+c)=(+ac)$,

then $(+a)\times(-d)=(-ad)$,

$(-b)\times(+c)=(-bc)$,

$(-b)\times(-d)=(+bd)$.

Hence, the law of signs in multiplication,

Like **signs give** ***plus*****;** ***unlike*** **signs give** ***minus.***

The product of more than two scalar factors, each preceded by the sign $-$, will be *positive* or *negative*, according as the number of such factors is *even* or *odd*.

56. Index Law. *The product of two or more powers of any number is that number with an exponent equal to the sum of the exponents of the several factors.*

$$\begin{aligned}\text{For,}\quad a^m \times a^n &= (aaa \cdots \text{to } m \text{ factors})(aaa \cdots \text{to } n \text{ factors})\\ &= aaaaaa \cdots \text{to } (m+n) \text{ factors}\\ &= a^{m+n}.\end{aligned}$$

Similarly for more than two factors.

57. Monomials. The product of numerical factors is a new number in which no trace of the original factors is found.

Thus, $4 \times 9 = 36$.

But the product of literal factors is expressed by writing them one after the other.

Thus, the product of ab and cd is expressed by $abcd$, and generally the product of ae^m and be^n is abe^{m+n}, for $ae^m \times be^n = a \times b \times e^m \times e^n$ by the commutative law, $= a \times b \times e^{m+n}$ by the index law.

Hence, **to find the product of two monomials,**

Multiply the coefficients; affix all the literal parts, each with an exponent which is the sum of its exponents in the separate factors; prefix the sign $+$ if the signs of the monomials are alike, the sign $-$ if they are unlike.

58. Polynomials. To multiply a polynomial by a monomial, the distributive law, § 54, p. 24, may be applied, giving as rule:

Multiply every term of the polynomial by the monomial multiplier, observing the law of signs, § 55, p. 25.

To multiply a polynomial by a polynomial, we apply the distributive law, as in § 54, p. 24, and obtain as rule:

Multiply every term in the multiplicand by every term in the multiplier, observing the law of signs, § 55, p. 25.

59. In multiplying polynomials it is a convenient arrangement to write the multiplier under the multiplicand, and place like terms of the partial products in columns.

(1) Multiply $5a - 6b$ by $3a - 4b$.

$$\begin{array}{rrr}
5a & -\ 6b & \\
3a & -\ 4b & \\
\hline
15a^2 & -\ 18ab & \\
 & -\ 20ab & +\ 24b^2 \\
\hline
15a^2 & -\ 38ab & +\ 24b^2
\end{array}$$

(2) Multiply $a^2 + b^2 + c^2 - ab - bc - ac$ by $a + b + c$.

Arrange according to descending powers of a.

$$\begin{array}{llllllllll}
a^2 & -\ ab & -\ ac & +\ b^2 & -\ bc & +\ c^2 & & & & \\
a & +\ b & +\ c & & & & & & & \\
\hline
a^3 & -\ a^2b & -\ a^2c & +\ ab^2 & -\ abc & +\ ac^2 & & & & \\
 & +\ a^2b & & -\ ab^2 & -\ abc & & +\ b^3 & -\ b^2c & +\ bc^2 & \\
 & & +\ a^2c & & -\ abc & -\ ac^2 & & +\ b^2c & -\ bc^2 & +\ c^3 \\
\hline
a^3 & & & & -\ 3abc & & +\ b^3 & & & +\ c^3
\end{array}$$

Observe that, with a view to bringing like terms of the partial products into columns, the terms of the multiplicand and of the multiplier are arranged in the *same order.*

60. Detached Coefficients. In multiplying two polynomials that involve but one letter, or are homogeneous (§ 23, p. 10) and involve but two letters, we shall save much labor if we write only the coefficients.

(1) Multiply $2x^3 + 4x + 7$ by $x^2 - 3x + 4$.

Since the x^2 term in the first expression is missing, we supply a zero coefficient. The work is as follows:

$$\begin{array}{rrrrrr}
2 & +\ 0 & +\ 4 & +\ 7 & & \\
1 & -\ 3 & +\ 4 & & & \\
\hline
2 & +\ 0 & +\ 4 & +\ 7 & & \\
 & -\ 6 & -\ 0 & -\ 12 & -\ 21 & \\
 & & +\ 8 & +\ 0 & +\ 16 & +\ 28 \\
\hline
2 & -\ 6 & +\ 12 & -\ 5 & -\ 5 & +\ 28
\end{array}$$

Writing in the powers of x, the product is

$$2x^5 - 6x^4 + 12x^3 - 5x^2 - 5x + 28.$$

(2) Multiply $a^3 + 2ax^2 - 9x^3 + 4a^2x$ by $x^2 - 2ax - a^2$.

Arranging by descending powers of x we have

$$-9x^3 + 2ax^2 + 4a^2x + a^3 \text{ and } x^2 - 2ax - a^2.$$

The work is as follows:

$$\begin{array}{rrrrrr}
-9 & +2 & +4 & +1 & & \\
1 & -2 & -1 & & & \\
\hline
-9 & +2 & +4 & +1 & & \\
 & +18 & -4 & -8 & -2 & \\
 & & +9 & -2 & -4 & -1 \\
\hline
-9 & +20 & +9 & -9 & -6 & -1
\end{array}$$

Hence, the product is $-9x^5 + 20ax^4 + 9a^2x^3 - 9a^3x^2 - 6a^4x - a^5$.

61. Special Cases. The following products are of great importance, and should be carefully remembered:

$$(a - b)^2 = a^2 - 2ab + b^2;$$

$$(a + b)^2 = a^2 + 2ab + b^2;$$

$$(a + b)(a - b) = a^2 - b^2;$$

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

The *square* of any polynomial may be immediately written by the following rule:

Add together the squares of the several terms and twice the product of each term into each of the terms that follow it.

Also, $$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3.$$

The double sign $\pm$ is read *plus or minus*, and signifies the sum or the difference of the numbers between which it is placed.

62. Again, consider the product

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

The coefficient of x is the *algebraic sum* of a and b; the third term is the *product* of a and b.

Thus,
$$(x + 3)(x + 7) = x^2 + 10x + 21;$$
$$(x - 3)(x + 7) = x^2 + 4x - 21;$$
$$(x + 3)(x - 7) = x^2 - 4x - 21;$$
$$(x - 3)(x - 7) = x^2 - 10x + 21.$$

Exercise 3

Find the product of:

1. $3x + 2y$ and $4x - 5y$.
2. $2x^2 - 5$ and $4x + 3$.
3. $2x^2 + 4x - 3$ and $2x^2 + 3x - 4$.
4. $x^4 + 2x^2 + 4$ and $x^4 - 2x^2 + 4$.
5. $x^2 + 2xy - 3y^2$ and $x^2 - 5xy + 4y^2$.
6. $9x^2 + 3xy + y^2 - 6x + 2y + 4$ and $3x - y + 2$.
7. $11a^3 + 4b^3 - 4ab(a - 4b)$ and $a^2(b + 3a) - 4b^2(a + b)$.
8. $(a + b)^2 + (a - b)^2$ and $(a + b)^2 - (a - b)^2$.
9. $x - 2y + 3z$ and $x - 2y + 3z$.
10. $x^3 + 2x^2 - 4x - 1$ and $x^3 + 2x^2 - 4x - 1$.
11. $39d^{x+y-1} - 54d^{x-2y+1} + 60d^{x+3y}$ and $30d^{2-x+2y}$.
12. $24x^{m+2n-1} - 42x^{2m-3n+2} + 25x^{2n+3m-2}$ and $25x^{2-m-2n}$.
13. $a^p - 3a^{p-1} + 4a^{p-2} - 6a^{p-3} + 5a^{p-4}$ and $2a^3 - a^2 + a$.
14. $a^{2n+1} - a^{n+1} - a^n + a^{n-1}$ and $a^{n+2} - a^2 - a + 1$.
15. $a^p + 3a^{p-2} - 2a^{p-1}$ and $2a^{p+1} + a^{p+2} - 3a^p$.

DIVISION

63. Division is the operation by which, when a product and one of its factors are given, the other factor is determined, *the given factor not being zero.* In symbols: Division is the operation symbolized by $a \div b$, or $\frac{a}{b}$, or $a : b$ such that

$$(a \div b) \times b = a, \text{ or } \frac{a}{b} \times b = a, \text{ or } (a : b) \times b = a;$$

and, as a consequence of law II, p. 23, such that

$$b \times (a \div b) = a, \text{ or } b \times \frac{a}{b} = a, \text{ or } b \times (a : b) = a;$$

in which a may have *any* value, and b any value *except zero.*

In this operation the product is called the **dividend**; the given factor the **divisor**; and the required factor the **quotient**.

64. The laws of division are not fundamental but are derived from this definition combined with the laws of multiplication. They are:

i. *If equals are divided by equals, the quotients are equal.*

ii. *Dividing a factor by any number divides the product by that number.*

iii. *Multiplying the dividend by any number multiplies the quotient by that number.*

iv. *Dividing the dividend by any number divides the quotient by that number.*

v. *Multiplying the divisor by any number divides the quotient by that number.*

vi. *Dividing the divisor by any number multiplies the quotient by that number.*

vii. *If the quotient is equal to the dividend, the divisor is unity.*

viii. *Dividing all the terms of a polynomial by any number divides the polynomial by that number.*

65. These laws expressed in algebraic notation are:

Suppose a, b, c, m, and n have each one and *only one* value, zero included as a possible value for a, b, and c but not for m and n.

If $$a = c \text{ and } m = n,$$

then $$a \div m = c \div n. \qquad \text{(i)}$$

$$(a \div m) \times c = (a \times c) \div m,$$

and $$a \times (c \div m) = (a \times c) \div m. \qquad \text{(ii)}$$

$$(a \times c) \div m = (a \div m) \times c. \qquad \text{(iii)}$$

$$(a \div n) \div m = (a \div m) \div n. \qquad \text{(iv)}$$

$$a \div (m \times n) = (a \div m) \div n. \qquad \text{(v)}$$

$$a \div (m \div n) = (a \div m) \times n. \qquad \text{(vi)}$$

If $$a \div m = a, \text{ then } m = 1. \qquad \text{(vii)}$$

$$a \div m + b \div m - c \div m = (a + b - c) \div m,$$

or $$\frac{a}{m} + \frac{b}{m} - \frac{c}{m} = \frac{a+b-c}{m}. \qquad \text{(viii)}$$

The fundamental law VI, p. 24, and law viii of this section are both included in the single proposition:

Multiplication is *completely* distributive relative to addition.

66. Since $a \times b = +ab$, $\quad (-a) \times b = -ab$,

$$\therefore \frac{ab}{b} = a; \qquad \therefore \frac{-ab}{+b} = -a;$$

$$a \times (-b) = -ab, \quad (-a) \times (-b) = +ab,$$

$$\therefore \frac{-ab}{-b} = +a; \qquad \therefore \frac{+ab}{-b} = -a.$$

Consequently, the quotient is *positive* when the dividend and divisor have *like* signs.

The quotient is *negative* when the dividend and divisor have *unlike* signs.

67. Monomials. To divide one monomial by another,

Write the dividend over the divisor with a line between them; if the expressions have common factors, remove the common factors.

Thus, $$\frac{25\,abx}{10\,bcx} = \frac{5\,a}{2\,c}; \qquad \frac{36\,bcx}{30\,abc} = \frac{6\,x}{5\,a}.$$

Again, $$\frac{a^5}{a^2} = \frac{aaaaa}{aa} = aaa = a^3;$$

$$\frac{a^2}{a^5} = \frac{aa}{aaaaa} = \frac{1}{aaa} = \frac{1}{a^3}.$$

In general,
$$\frac{a^m}{a^n} = \frac{aaa \cdots \text{to } m \text{ factors}}{aaa \cdots \text{to } n \text{ factors}}$$
$$= aaa \cdots \text{to } m - n \text{ factors (if } m > n\text{)},$$
or
$$= \frac{1}{aaa \cdots \text{to } n - m \text{ factors}} \text{ (if } n > m\text{)}.$$

Hence, if a power of a number is divided by a *lower* power of the same number,

The quotient is that power of the number of which the exponent is the exponent of the dividend diminished by that of the divisor.

If any power of a number is divided by a *higher* power of the same number,

The quotient is expressed by 1 *divided by that power of the number of which the exponent is the exponent of the divisor diminished by that of the dividend.*

68. Polynomials by Monomials. When the divisor is a monomial and the dividend a polynomial,

Divide each term of the dividend by the monomial divisor; the required quotient is the sum of the partial quotients.

For since
$$(a + b - c) \times m = ma + mb - mc,$$
$$\therefore (ma + mb - mc) \div m = a + b - c.$$

The *signs* are determined by § 66, p. 31.

69. Division of Polynomials by Polynomials.

If the divisor (one factor) is $a + b + c$,
and the quotient (other factor) is $n + p + q$,

then the dividend (product) is
$$\begin{cases} an + bn + cn \\ + ap + bp + cp \\ + aq + bq + cq. \end{cases}$$

The first term of the dividend is an, the product of a, the first term of the divisor, by n, the first term of the quotient.

The first term n of the quotient is therefore found by dividing an, the first term of the dividend, by a, the first term of the divisor.

If the partial product formed by multiplying the entire divisor by n is subtracted from the dividend, ap, the first term of the remainder, is the product of a, the first term of the divisor, by p, the second term of the quotient. Hence, the second term of the quotient is obtained by dividing the first term of the remainder by the first term of the divisor; and so on.

Therefore, **to divide one polynomial by another,**

Divide the first term of the dividend by the first term of the divisor.

Write the result as the first term of the quotient.

Multiply all the terms of the divisor by the first term of the quotient.

Subtract the product from the dividend.

If there is a remainder, consider it as a new dividend and proceed as before.

It is of great importance to arrange both dividend and divisor according to the ascending or the descending *powers of some common letter*, and to *keep this order* throughout the operation.

(1) Divide

$$22\,a^2b^2 + 15\,b^4 + 3\,a^4 - 10\,a^3b - 22\,ab^3 \text{ by } a^2 + 3\,b^2 - 2\,ab.$$

Arrange the dividend and divisor according to the descending powers of a and divide.

$$\begin{array}{rrrrr|l}
3a^4 & -10a^3b & +22a^2b^2 & -22ab^3 & +15b^4 & \;a^2-2ab+3b^2 \\ \cline{6-6}
3a^4 & -6a^3b & +9a^2b^2 & & & \;3a^2-4ab+5b^2 \\ \cline{1-3}
 & -4a^3b & +13a^2b^2 & -22ab^3 & & \\
 & -4a^3b & +8a^2b^2 & -12ab^3 & & \\ \cline{2-4}
 & & 5a^2b^2 & -10ab^3 & +15b^4 & \\
 & & 5a^2b^2 & -10ab^3 & +15b^4 & \\ \cline{3-5}
\end{array}$$

The operation of division may be shortened in some cases by the use of parentheses.

(2) Divide

$$x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc \text{ by } x + b.$$

$$\begin{array}{l|l}
x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc & x + b \\ \cline{2-2}
\underline{x^3 + (\quad + b \quad)x^2} & x^2 + (a + c)x + ac \\
\qquad (a \quad + c)x^2 + (ab + ac + bc)x & \\
\qquad \underline{(a \quad + c)x^2 + (ab \quad + bc)x} & \\
\qquad\qquad\qquad\qquad acx \quad + abc & \\
\qquad\qquad\qquad\qquad \underline{acx \quad + abc} &
\end{array}$$

70. Detached Coefficients. In division, as in multiplication, it is convenient to use only the coefficients when the dividend and divisor are expressions involving but one letter, or homogeneous expressions involving but two letters.

Thus, the work of Example (1), § 69, may be arranged as follows:

$$\begin{array}{l|l}
3 - 10 + 22 - 22 + 15 & 1 - 2 + 3 \\ \cline{2-2}
\underline{3 - \ \ 6 + \ \ 9} & 3 - 4 + 5 \\
\quad - \ \ 4 + 13 - 22 & \\
\quad \underline{- \ \ 4 + \ \ 8 - 12} & \\
\qquad\qquad 5 - 10 + 15 & \\
\qquad\qquad \underline{5 - 10 + 15} &
\end{array}$$

The quotient is $3a^2 - 4ab + 5b^2$.

71. Special Cases. There are some cases in division which occur so often in algebraic operations that they should be carefully noticed and remembered.

The student may easily verify the following results:

$$(1) \quad \frac{a^3 - b^3}{a - b} = a^2 + ab + b^2.$$

$$(2) \quad \frac{a^5 - b^5}{a - b} = a^4 + a^3b + a^2b^2 + ab^3 + b^4.$$

In general, the difference of two like powers of any two numbers is divisible by the difference of the numbers.

(3) $\dfrac{\mathbf{a^3 + b^3}}{\mathbf{a + b}} = \mathbf{a^2 - ab + b^2}.$

(4) $\dfrac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4.$

In general, the sum of two like *odd* powers of two numbers is divisible by the sum of the numbers.

Compare (3) and (4) with (1) and (2).

(5) $\dfrac{\mathbf{x^2 - y^2}}{\mathbf{x - y}} = \mathbf{x + y}.$ (7) $\dfrac{x^4 - y^4}{x - y} = x^3 + x^2y + xy^2 + y^3.$

(6) $\dfrac{\mathbf{x^2 - y^2}}{\mathbf{x + y}} = \mathbf{x - y}.$ (8) $\dfrac{x^4 - y^4}{x + y} = x^3 - x^2y + xy^2 - y^3.$

In general, the *difference* of two like *even* powers of two numbers is divisible by the difference and also by the sum of the numbers.

The *sum* of two like *even* powers of two numbers is not divisible by either the sum or the difference of the numbers.

But when the exponent of each of the two like powers is composed of an *odd* and an *even* factor, the sum of the given powers is divisible by the sum of the powers expressed by the even factor.

Thus, $x^6 + y^6$ is not divisible by $x + y$, or by $x - y$, but is divisible by $x^2 + y^2$.

The quotient may be found as in (3) and (4).

A factor of $x^n - y^n$ can always be found; and a factor of $x^n + y^n$ can always be found *unless* n *is a power of* 2.

Thus, factors of $x^2 + y^2$, $x^4 + y^4$, $x^8 + y^8$, etc., cannot be found.

Exercise 4

Divide:

1. $(6\,a^2b^3c \times 35\,a^2b^5c^4)$ by $(21\,a^3b^9c^6 \times 2\,a^2c^3)$.
2. $39\,a^3x^2 + 24\,a^4x^3 + 42\,a^2x^3 + 27\,a^4x^2$ by $6\,a^2x^2$.
3. $35\,x^3 + 94\,ax^2 + 52\,a^2x + 8\,a^3$ by $5\,x + 2\,a$.

4. $x^3 - 5ax^2 - a^2x + 14a^3$ by $x^2 - 3ax - 7a^2$.

5. $81x^4 + 36x^2y^2 + 16y^4$ by $9x^2 - 6xy + 4y^2$.

6. $x^4 + b^4 - a^2x^2 + 2b^2x^2$ by $x^2 + b^2 + ax$.

7. $a^2 - 2b^2 - 3c^2 + ab + 2ac + 7bc$ by $a - b + 3c$.

8. $4x^4 - 5x^2y^2 - 8x^2 - 4y^2 + 4 + y^4$
by $y^2 + 2x^2 - 2 - 3xy$.

9. $2a^{m+1} - 2a^{n+1} - a^{m+n} + a^{2n}$ by $a^n - 2a$.

10. $625x^4 - 81y^4$ by $5x - 3y$.

11. $x^{3n} + y^{3n}$ by $x^n + y^n$.

12. $\dfrac{27a^3}{125} - \dfrac{b^3}{64}$ by $\dfrac{3a}{5} - \dfrac{b}{4}$.

13. $(a + 2b)^3 + (b - 3c)^3$ by $a + 3(b - c)$.

14. $a^m - a^{m+1} + 37a^{m+3} - 55a^{m+4} + 50a^{m+5}$
by $1 - 3a + 10a^2$.

15. $4h^{x+1} - 30h^x + 19h^{x-1} + 5h^{x-2} + 9h^{x-4}$
by $h^{x-3} - 7h^{x-4} + 2h^{x-5} - 3h^{x-6}$.

16. $6x^{m-n+2} + x^{m-n+1} - 22x^{m-n} + 19x^{m-n-1} - 4x^{m-n-2}$
by $3x^{3-n} - 4x^{2-n} + x^{1-n}$.

72. Summary. The four elementary operations of Algebra are performed subject to I, The Law of Uniformity; II, The Associative Law; III, The Commutative Law; and IV, The Distributive Law. The meanings of these laws have been explained as occasion arose; we here sum up the whole in brief review.

1. From the number a and the number b there is determined by addition a definite number c which is expressed thus:

$$a + b = c, \qquad \text{or} \qquad c = a + b.$$

2. There is a determinate number which we name *zero* and denote by 0, such that for every number a we have simultaneously

$$a + 0 = a, \qquad \text{and} \qquad 0 + a = a.$$

3. There is a number which we name *infinity* and denote by ∞, such that for every number a we have simultaneously

$$a + \infty = \infty, \qquad \text{and} \qquad \infty + a = \infty.$$

4. If a and b denote given numbers, a not being infinity, there always exists one and only one number x and also one and only one number y, such that we have respectively

$$a + x = b, \qquad \text{and} \qquad y + a = b.$$

5. From the number a and the number b there is determined by multiplication a definite number c which is expressed thus:

$$ab = c, \qquad \text{or} \qquad c = ab.$$

6. There is a determinate number which we name *unity* and denote by 1, such that for every number a we have simultaneously

$$a \times 1 = a, \qquad \text{and} \qquad 1 \times a = a.$$

7. For every finite number a we have simultaneously

$$a \times 0 = 0, \qquad \text{and} \qquad 0 \times a = 0.$$

8. If a and b denote given numbers, a not being zero and b not being infinity, there always exists one and only one number x and also one and only one number y, such that we have respectively

$$ax = b, \qquad \text{and} \qquad ya = b.$$

If a, b, and c denote any numbers whatever, the following laws of calculation always hold true:

9. $$a + (b + c) = (a + b) + c.$$

10. $$a + b = b + a.$$

11. $$a(bc) = (ab)c.$$

12. $$ab = ba.$$

13. $$a(b + c) = ab + ac.$$

CHAPTER III

FACTORS

73. Rational Integral Expressions. An expression is *rational* if none of its terms contains indicated roots.

An expression is **integral** if none of its terms contains other than positive integral powers.

74. Factors of Rational and Integral Expressions. By factors of *a rational and integral expression* we mean rational and integral expressions that will exactly divide the given expression.

75. Factors of Monomials. The factors of a monomial may be found by inspection.

76. Factors of Polynomials. The *form* of a polynomial that can be resolved into factors often suggests the process of finding the factors.

77. When the terms have a common monomial factor.

Resolve into factors $6a^3 + 4a^2 + 8a$.

Since 2 and a are factors of each term, we have

$$6a^3 + 4a^2 + 8a = 2a(3a^2 + 2a + 4).$$

Hence, the required factors are 2, a, and $3a^2 + 2a + 4$.

78. When the terms can be grouped so as to show a common compound factor.

Resolve into factors $ac - ad - bc + bd$.

$$\begin{aligned} ac - ad - bc + bd &= (ac - ad) - (bc - bd) \\ &= a(c - d) - b(c - d) \\ &= (a - b)(c - d). \end{aligned}$$

Hence, the required factors are $a - b$ and $c - d$.

79. Square Roots. If an expression can be resolved into two equal factors, one of the equal factors is called the **square root** of the expression (§ 19, p. 7).

Thus, $$16\,x^6y^2 = 4\,x^3y \times 4\,x^3y.$$

Hence, $4\,x^3y$ is the square root of $16\,x^6y^2$.

The square root of a positive number may be either positive or negative; for

$$a^2 = a \times a, \quad \text{and} \quad a^2 = (-a) \times (-a).$$

Throughout this chapter the *positive* square root only will be considered.

80. When a Trinomial is a Perfect Square. A trinomial is a perfect square if the first and last terms are perfect squares and positive, and the middle term is twice the product of the square roots of the first and last terms (§ 61, p. 28).

Thus, $16\,a^2 - 24\,ab + 9\,b^2$ is a perfect square.

To extract the square root of a trinomial that is a perfect square,

Extract the square root of the first term and of the last term and connect these square roots by the sign of the middle term.

Resolve into factors $x^2 - 18\,x + 81$.

$$x^2 - 18\,x + 81 = (x-9)(x-9) = (x-9)^2.$$

Hence, the required factors are $x - 9$ and $x - 9$.

81. When a Binomial is the Difference of Two Squares. The difference of two squares is the product of two factors which may be determined as follows:

Extract the square root of the first number and the square root of the second number.

The sum of these roots will form the first factor.

The difference of these roots will form the second factor.

Thus, (1) $a^2 - b^2 = (a+b)(a-b)$;

(2) $(a-b)^2 - (c-d)^2 = \{(a-b)+(c-d)\}\{(a-b)-(c-d)\}$
$= \{a-b+c-d\}\{a-b-c+d\}$.

The terms of an expression may often be arranged so as to form the difference of two squares, and the expression can then be resolved into factors.

Thus, $a^2 + b^2 - c^2 - d^2 + 2\,ab + 2\,cd$

$$\begin{aligned} &= a^2 + 2\,ab + b^2 - c^2 + 2\,cd - d^2 \\ &= (a^2 + 2\,ab + b^2) - (c^2 - 2\,cd + d^2) \\ &= (a + b)^2 - (c - d)^2 \\ &= \{(a + b) + (c - d)\}\{(a + b) - (c - d)\} \\ &= \{a + b + c - d\}\{a + b - c + d\}. \end{aligned}$$

A trinomial in the form $a^4 + a^2b^2 + b^4$ can be written as the difference of two squares and resolved into factors.

Thus,
$$\begin{aligned} x^4 + x^2y^2 + y^4 &= (x^4 + 2\,x^2y^2 + y^4) - x^2y^2 \\ &= (x^2 + y^2)^2 - (xy)^2 \\ &= (x^2 + y^2 + xy)(x^2 + y^2 - xy) \\ &= (x^2 + xy + y^2)(x^2 - xy + y^2). \end{aligned}$$

A binomial in the form $x^4 + 4\,y^4$ can be written as the difference of two squares and resolved into two factors.

Thus,
$$\begin{aligned} 1 + 4\,y^4 &= (1 + 4\,y^2 + 4\,y^4) - 4\,y^2 \\ &= (1 + 2\,y^2)^2 - (2\,y)^2 \\ &= (1 + 2\,y + 2\,y^2)(1 - 2\,y + 2\,y^2). \end{aligned}$$

Many expressions may be resolved into three or more factors.

Thus,
$$\begin{aligned} x^{16} - y^{16} &= (x^8 + y^8)(x^8 - y^8) \\ &= (x^8 + y^8)(x^4 + y^4)(x^4 - y^4) \\ &= (x^8 + y^8)(x^4 + y^4)(x^2 + y^2)(x^2 - y^2) \\ &= (x^8 + y^8)(x^4 + y^4)(x^2 + y^2)(x + y)(x - y). \end{aligned}$$

82. A Trinomial of the Form $x^2 + ax + b$, where a is the *algebraic sum* of two numbers and is either positive or negative, and b is the *product* of these two numbers and is either positive or negative, can be resolved into factors.

Since $\qquad (x + 5)(x + 3) = x^2 + 8\,x + 15,$

the factors of $x^2 + 8\,x + 15$ are $x + 5$ and $x + 3$.

Since $\qquad (x + 5)(x - 3) = x^2 + 2\,x - 15,$

the factors of $x^2 + 2\,x - 15$ are $x + 5$ and $x - 3$.

Hence, if a trinomial of the form $x^2 + ax + b$ is such an expression that it can be resolved into two binomial factors, the first term of each factor will be x; the second terms of the factors will be two numbers *whose product is b*, the last term of the trinomial, and *whose algebraic sum is a*, the coefficient of x in the middle term of the trinomial.

(1) Resolve into factors $x^2 + 11x + 30$.

We are required to find two numbers whose product is 30 and whose sum is 11.

Two numbers whose product is 30 are 1 and 30, 2 and 15, 3 and 10, 5 and 6; and the sum of the last two numbers is 11.

Hence, $$x^2 + 11x + 30 = (x + 5)(x + 6).$$

(2) Resolve into factors $x^2 - 7x + 12$.

We are required to find two numbers whose product is 12 and whose algebraic sum is -7.

Since the product is $+12$, the two numbers are *both positive* or *both negative;* and since their sum is -7, they must both be negative.

Two negative numbers whose product is 12 are -12 and -1, -6 and -2, -4 and -3; and the sum of the last two numbers is -7.

Hence, $$x^2 - 7x + 12 = (x - 4)(x - 3).$$

(3) Resolve into factors $x^2 + 2x - 24$.

We are required to find two numbers whose product is -24 and whose algebraic sum is 2.

Since the product is -24, one of the numbers is positive and the other negative; and since their sum is $+2$, the larger number is positive.

Two numbers whose product is -24, and the larger number positive, are 24 and -1, 12 and -2, 8 and -3, 6 and -4; and the sum of the last two numbers is $+2$.

Hence, $$x^2 + 2x - 24 = (x + 6)(x - 4).$$

(4) Resolve into factors $x^2 - 3x - 18$.

We are required to find two numbers whose product is -18 and whose algebraic sum is -3.

Since the product is -18, one of the numbers is positive and the other negative; and since their sum is -3, the larger number is negative.

Two numbers whose product is -18, and the larger number negative, are -18 and 1, -9 and 2, -6 and 3; and the sum of the last two numbers is -3.

Hence, $$x^2 - 3x - 18 = (x - 6)(x + 3).$$

Therefore, in general,

$$x^2 + (a + b)x + ab = (x + a)(x + b)$$

whatever the *values* of a and b.

83. When a Trinomial has the Form $ax^2 + bx + c$.

(1) Resolve into factors $8x^2 - 22x - 21$.

Multiply by 8, the coefficient of x^2, and write the result in the following form:

$$(8x)^2 - 22 \times 8x - 168.$$

Put z for $8x$, $$z^2 - 22z - 168.$$

Resolve this expression into two binomial factors,

$$(z - 28)(z + 6). \qquad (\S\ 82,\ \text{p. } 40)$$

Since we have multiplied by 8, and put z for $8x$, we must reverse this process. Hence, put $8x$ for z and divide by 8, and we have

$$\frac{(8x - 28)(8x + 6)}{8}.$$

As 4 is a factor of $(8x - 28)$, and 2 is a factor of $(8x + 6)$, we divide by 8 by dividing the first factor by 4 and the second factor by 2.

Then, $$\frac{(8x - 28)(8x + 6)}{4 \times 2} = (2x - 7)(4x + 3).$$

(2) Resolve into factors $24x^2 - 70xy - 75y^2$.

Multiply by 24, $(24x)^2 - 70y \times 24x - 1800y^2$.
Put z for $24x$, $z^2 - 70yz - 1800y^2$.
Resolve into factors, $(z - 90y)(z + 20y)$. (§ 82, p. 40)
Put $24x$ for z, $(24x - 90y)(24x + 20y)$.
Divide by 6×4, $(4x - 15y)(6x + 5y)$.

84. When a Binomial is the Sum or the Difference of Two Cubes.

From § 71, p. 34, $$\frac{a^3 + b^3}{a + b} = a^2 - ab + b^2;$$

and $$\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2.$$

$$\therefore a^3 + b^3 = (a + b)(a^2 - ab + b^2);$$

and
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

In like manner we can resolve into factors any expression which can be written as the sum or the difference of two cubes.

(1) Resolve into factors $8a^3 + 27b^6$.

$$\begin{aligned} 8a^3 + 27b^6 &= (2a)^3 + (3b^2)^3 \\ &= [2a + 3b^2][(2a)^2 - (2a)(3b^2) + (3b^2)^2] \\ &= (2a + 3b^2)(4a^2 - 6ab^2 + 9b^4). \end{aligned}$$

85. When a Polynomial is the Product of Two Trinomials. The following method is convenient for resolving a polynomial into its trinomial factors:

Find the factors of $2x^2 - 5xy + 2y^2 + 7xz - 5yz + 3z^2$.

1. Reject the terms that contain z.
2. Reject the terms that contain y.
3. Reject the terms that contain x.

Factor the expression that remains in each case.

1. $2x^2 - 5xy + 2y^2 = (x - 2y)(2x - y)$.
2. $2x^2 + 7xz + 3z^2 = (x + 3z)(2x + z)$.
3. $2y^2 - 5yz + 3z^2 = (2y - 3z)(y - z)$.

Arrange these three pairs of factors in two rows of three factors each, so that any two factors of each row may have a *common term including the sign.*

Thus,
1. $x - 2y,\ x + 3z,\ -2y + 3z;$
2. $2x - y,\ 2x + z,\ -y + z.$

From the first row, select the *terms common to two factors* for one trinomial factor:

$$x - 2y + 3z.$$

From the second row, select the *terms common to two factors* for the other trinomial factor:

$$2x - y + z.$$

Then,

$$2x^2 - 5xy + 2y^2 + 7xz - 5yz + 3z^2 = (x - 2y + 3z)(2x - y + z).$$

When a factor obtained from the first three terms is also a factor of the remaining terms, the expression is easily factored.

Thus, $$\begin{aligned} x^2 - 3xy + 2y^2 - 3x + 6y &= (x - 2y)(x - y) - 3(x - 2y) \\ &= (x - 2y)(x - y - 3). \end{aligned}$$

THEORY OF DIVISORS

86. Theorem. *The expression* x − y *is an exact divisor of* $x^n - y^n$ *when* n *is any positive integer.*

Since $$-x^{n-1}y + x^{n-1}y = 0, \qquad (\S\ 23,\ p.\ 9)$$
$$x^n - y^n = x^n - x^{n-1}y + x^{n-1}y - y^n.$$

Taking out x^{n-1} from the first two terms of the right side, and y from the last two terms, we have

$$x^n - y^n = x^{n-1}(x-y) + y(x^{n-1} - y^{n-1}).$$

Now $x - y$ is an exact divisor of the right side, if it is an exact divisor of $x^{n-1} - y^{n-1}$; and if $x - y$ is an exact divisor of the right side, it is an exact divisor of the left side; that is, $x - y$ is an exact divisor of $x^n - y^n$ *if* it is an exact divisor of $x^{n-1} - y^{n-1}$.

Therefore, *if* x − y *is an exact divisor of the difference of any two like powers of* x − y, *it is an exact divisor of the difference of the next higher powers of* x − y.

But $x - y$ *is* an exact divisor of $x^3 - y^3$ (§ 71), therefore it is an exact divisor of $x^4 - y^4$; and since it is an exact divisor of $x^4 - y^4$, it is an exact divisor of $x^5 - y^5$; and so on, indefinitely.

The method employed in proving this Theorem is called **Proof by Mathematical Induction.**

87. The Factor Theorem. *If a rational and integral expression in* x *vanishes, that is, becomes equal to* 0, *when* r *is put for* x, *then* x − r *is an exact divisor of the expression.*

Given $$ax^n + bx^{n-1} + \cdots + hx + k. \qquad [1]$$

By supposition, $$ar^n + br^{n-1} + \cdots + hr + k = 0. \qquad [2]$$

By subtracting [2] from [1], the given expression assumes the form

$$a(x^n - r^n) + b(x^{n-1} - r^{n-1}) + \cdots + h(x - r).$$

But $x - r$ is an exact divisor of $x^n - r^n$, $x^{n-1} - r^{n-1}$, and so on. (§ 86)

Therefore, $x - r$ is an exact divisor of the given expression.

Note. If $x - r$ is an exact divisor of the given expression, r is an exact divisor of k; for k, the last term of the dividend, is equal to r, the last term of the divisor, multiplied by the last term of the quotient.

Therefore, in searching for numerical values of x that will make the given expression vanish, only exact divisors of the last term of the expression need be tried.

(1) Resolve into factors $x^3 + 3x^2 - 13x - 15$.

The exact divisors of -15 are $1, -1, 3, -3, 5, -5, 15, -15$.

If we put 1 for x in $x^3 + 3x^2 - 13x - 15$, the expression does not vanish. If we put -1 for x, the expression vanishes.

Therefore, $x - (-1)$, that is, $x + 1$, is a factor.

Divide the expression by $x + 1$, and we have

$$\begin{aligned} x^3 + 3x^2 - 13x - 15 &= (x+1)(x^2 + 2x - 15) \\ &= (x+1)(x-3)(x+5). \end{aligned}$$

(2) Resolve into factors $x^3 - 26x - 5$.

By trial we find that the only exact divisor of -5 that makes the expression vanish is -5.

Therefore, divide by $x + 5$, and we have

$$x^3 - 26x - 5 = (x+5)(x^2 - 5x - 1).$$

As neither $+1$ nor -1, the exact divisors of -1, will make $x^2 - 5x - 1$ vanish, this expression cannot be resolved into factors.

Exercise 5

Resolve into factors:

1. $9x^4 + 6x^3 + 3x^2 + 2x$.
2. $2a^4 - 3a^3b - 14a^2 + 21ab$.
3. $5x^3 + 15x^2y - 4xy^2 - 12y^3$.
4. $a^2x^3 - b^2xy^2 - a^2cx^2 + b^2cy^2$.
5. $x^2 + 8x + 7$.
6. $x^2 - 17x + 60$.
7. $x^2 + 7x - 18$.
8. $x^2 - 2x - 24$.
9. $9x^2 + 30x + 25$.
10. $16x^2 - 56x + 49$.
11. $x^2 + x - 72$.
12. $x^2 - 14x - 176$.
13. $81x^4 - 196x^2y^2$.
14. $729a^6 - x^6$.
15. $64x^7 + xy^6$.
16. $(x^2 - y^2)^2 - y^4$.
17. $(a^2 + 2b^2)^2 - a^2b^2$.
18. $(2x - 3y)^2 - (x - 2y)^2$.

19. $121x^4 - 286x^2y + 169y^2$.

20. $a^2 - 2ab + b^2 - x^2$.

21. $49a^4 - 15a^2b^2 + 121b^4$.

22. $a^2x^2 + 14abx + 33b^2$.

23. $x^2y^2 + 23xyz + 90z^2$.

24. $a^2 + a - 132$.

25. $8a^2 + 14ab - 15b^2$.

26. $6x^2 + 19xy - 7y^2$.

27. $11a^2 - 23ab + 2b^2$.

28. $a^4 + 64b^4$.

29. $2x^2 - 5xy + 2y^2 - xz - yz - z^2$.

30. $6x^2 - 13xy + 6y^2 + 12xz - 13yz + 6z^2$.

31. $2x^2 + 5xy - 3y^2 - 4xz + 2yz$.

32. $4x^3 - 12x^2 + 9x - 1$.

33. $x^3 + 9x^2 + 16x + 4$.

34. $x^3 + 5x^2 + 7x + 2$.

35. $2x^2 - 3xy + 4ax - 6ay$.

36. $x^3z^2 - 8y^3z^2 - 4x^3n^2 + 32y^3n^2$.

37. $x^3 - y^3 - (x^2 - y^2) - (x - y)^2$.

38. $x^4 + 2x^3 - 13x^2 - 38x - 24$.

39. $x^4 - 2(b^2 - c^2)x^2 + b^4 - 2b^2c^2 + c^4$.

40. $15x^2 - 7x - 2$.

41. $11x^2 - 54x + 63$.

42. $21x^2 + 26x - 15$.

43. $70x^2 - 27x - 9$.

44. $x^4 - 2abx^2 - a^4 - a^2b^2 - b^4$.

45. $5x^4 + 4x^3 - 20x - 125$.

46. $2x^4 - 5x^3 - x^2 - 2$.

47. $6x^4 - ax^3 - 2a^2x^2 + 3a^3x - 2a^4$.

HIGHEST COMMON FACTOR

88. **A common factor** of two or more integral and rational expressions is an expression that divides each of them without a remainder.

Two expressions that have no common factor except 1 are said to be *prime* to each other.

The **highest common factor** of two or more integral and rational expressions is an integral and rational expression of *highest degree* that will divide each of them without remainder.

For brevity, H.C.F. will be used for highest common factor.

Find the H.C.F. of

$$8a^2x^2 - 24a^2x + 16a^2 \text{ and } 12ax^2y - 12axy - 24ay.$$

$$\begin{aligned} 8a^2x^2 - 24a^2x + 16a^2 &= 8a^2(x^2 - 3x + 2) \\ &= 2^3a^2(x-1)(x-2); \\ 12ax^2y - 12axy - 24ay &= 12ay(x^2 - x - 2) \\ &= 2^2 \times 3ay(x+1)(x-2). \\ \therefore \text{the H.C.F.} &= 2^2a(x-2) = 4a(x-2). \end{aligned}$$

Hence, **to find the H.C.F. of two or more expressions,**

Resolve each expression into its prime factors.

The product of all the common factors, each factor being taken the least number of times it occurs in any of the given expressions, is the highest common factor required.

89. When it is required to find the H.C.F. of two or more expressions that cannot readily be resolved into their factors, the method to be employed is similar to that of the corresponding case in Arithmetic. And as that method consists in obtaining pairs of continually decreasing numbers which contain as a factor the H.C.F. required, so in Algebra, pairs of expressions of continually decreasing degrees are obtained, which contain as a factor the H.C.F. required.

90. This method is needed only to determine the *compound* factor of the H.C.F. *Simple* factors of the given expressions should be taken out, and the highest common factor of these factors reserved to be multiplied into the compound factor obtained.

Modifications of this method are sometimes needed.

(1) Find the H.C.F. of $4x^2-8x-5$ and $12x^2-4x-65$.

$$\begin{array}{r|l} 4x^2-8x-5) & 12x^2-4x-65(3 \\ & \underline{12x^2-24x-15} \\ & 20x-50 \end{array}$$

The first division ends here, for $20x$ is of lower degree than $4x^2$. But if $20x-50$ is made the divisor, $4x^2$ will not contain $20x$ an *integral* number of times.

Now, it is to be remembered that the H.C.F. sought is *contained in the remainder* $20x-50$, and that it is a *compound factor*. Hence, if the *simple factor* 10 is removed, the H.C.F. must still be contained in $2x-5$, and therefore the process may be continued with $2x-5$ for a divisor.

$$\begin{array}{r|l} 2x-5) & 4x^2-8x-5(2x+1 \\ & \underline{4x^2-10x} \\ & 2x-5 \\ & \underline{2x-5} \end{array}$$

$\therefore$ the H.C.F. is $2x-5$.

(2) Find the H.C.F. of

$$21x^3-4x^2-15x-2 \text{ and } 21x^3-32x^2-54x-7.$$

Writing only the coefficients (§ 70, p. 34), the work is as follows:

$$\begin{array}{r|l} 21-4-15-2) & 21-32-54-7(1 \\ & \underline{21-4-15-2} \\ & -28-39-5 \end{array}$$

The difficulty here cannot be obviated by *removing* a simple factor from the remainder, for $-28x^2-39x-5$ has no simple factor. In this case, the expression $21x^3-4x^2-15x-2$ must be *multiplied* by the simple factor 4 to make its first term divisible by $-28x^2$.

The *introduction* of such a factor can in no way affect the H.C.F. sought; for the H.C.F. contains only factors *common to the remainder and the last divisor*, and 4 is not a factor of the remainder.

The *signs* of all the terms of the remainder may be changed; for if an expression A is divisible by $-F$, it is divisible by $+F$.

The process then is continued by changing the signs of the remainder and multiplying the divisor by 4.

$$
\begin{array}{lr}
 & 28+39+5)\,84-16-60-8\,(3 \\
 & \underline{84+117+15} \\
 & -133-75-8 \\
\text{Multiply by } -4, & \underline{-4} \\
 & 532+300+32\,(19 \\
 & \underline{532+741+95} \\
\text{Divide by } -63, & -63)\,-441-63 \\
 & 7+1
\end{array}
$$

$$
\begin{array}{lr}
 & 7+1)\,28+39+5\,(4+5 \\
 & \underline{28+4} \\
 & 35+5 \\
\therefore \text{ the H.C.F. is } 7x+1. & \underline{35+5}
\end{array}
$$

In practice the work is most conveniently arranged as follows:

$$
\begin{array}{r|r|l}
21-4-15-2 & 21-32-54-7 & 1 \\
\underline{4} & \underline{21-4-15-2} & \\
84-16-60-8 & -1)\,-28-39-5 & \\
\underline{84+117+15} & 28+39+5 & 3+19 \\
-133-75-8 & \underline{28+4} & \\
\underline{-4} & 35+5 & \\
532+300+32 & \underline{35+5} & 4+5 \\
\underline{532+741+95} & & \\
-63)\,-441-63 & & \\
7+1 & &
\end{array}
$$

$\therefore$ the H.C.F. is $7x+1$.

91. In the examples worked out we have *assumed* that the divisor which is contained in the corresponding dividend without a remainder is the H.C.F. required.

The *proof* may be given as follows:

Let A and B stand for two expressions which have no monomial factors, and which are arranged according to the descending powers of a common letter, the degree of B being not higher than that of A in the common letter.

Let A be divided by B, and let Q stand for the quotient, and R for the remainder. Then, since the dividend is equal to the product of the divisor and quotient plus the remainder, we have

$$A = BQ + R. \qquad [1]$$

Since the remainder is equal to the dividend minus the product of the divisor and quotient, we have

$$R = A - BQ. \qquad [2]$$

Now, a factor of each of the terms of an expression is a factor of the expression. Hence, any common factor of B and R is a factor of $BQ + R$, and by [1] a factor of A. That is, a common factor of B and R is also a common factor of A and B.

Also, any common factor of A and B is a factor of $A - BQ$, and by [2] a factor of R. That is, a common factor of A and B is also a common factor of B and R.

Therefore, the common factors of A and B are *the same* as the common factors of B and R; and consequently the H.C.F. of A and B is *the same* as the H.C.F. of B and R.

The proof for each succeeding step in the process is precisely the same; so that the H.C.F. of *any* divisor and the corresponding dividend is the H.C.F. required.

If at any step there is no remainder, the divisor is a factor of the corresponding dividend, and is therefore the H.C.F. of itself and the corresponding dividend. Hence, *this divisor* is the H.C.F. required.

92. The methods of resolving expressions into factors, given in this chapter, often enable us to shorten the work of finding the H.C.F. required.

(1) Find the H.C.F. of

$$x^4 + 3x^3 + 12x - 16 \text{ and } x^3 - 13x + 12.$$

Both of these expressions vanish when 1 is put for x. Therefore, both are divisible by $x - 1$ (§ 87).

The first quotient is $x^3 + 4x^2 + 4x + 16 = (x^2 + 4)(x + 4)$.
The second quotient is $x^2 + x - 12 = (x - 3)(x + 4)$.
Therefore, the H.C.F. is $(x - 1)(x + 4)$.

(2) Find the H.C.F. of

$$2x^4 + 9x^3 + 14x + 3 \text{ and } 3x^4 + 14x^3 + 9x + 2.$$

$$\begin{array}{r|r|l}
2x^4 + 9x^3 + 14x + 3 & 3x^4 + 14x^3 + 9x + 2 & \\
 & 2 & \\
\hline
 & 6x^4 + 28x^3 + 18x + 4 & 3 \\
 & 6x^4 + 27x^3 + 42x + 9 & \\
\hline
 & x^3 - 24x - 5 &
\end{array}$$

The remainder, $x^3 - 24x - 5$, vanishes when 5 is put for x.
The quotient of $x^3 - 24x - 5$ divided by $x - 5$ is $x^2 + 5x + 1$.
Since 5 is not an exact divisor of 3, $x - 5$ is not a factor of $2x^4 + 9x^3 + 14x + 3$; but $x^2 + 5x + 1$ is found by trial to be a factor, and is, therefore, the H.C.F. required.

(3) Find the H.C.F. of

$$28x^2 + 39x + 5 \text{ and } 84x^3 - 16x^2 - 60x - 8.$$

By § 83, p. 42, the factors of $28x^2 + 39x + 5$ are $7x + 1$ and $4x + 5$.
The factor $7x + 1$ is the H.C.F. required.

(4) Find the H.C.F. of

$$2x^4 - 6x^3 - x^2 + 15x - 10;\ 4x^4 + 6x^3 - 4x^2 - 15x - 15.$$

$$\begin{array}{r|r|l}
2x^4 - 6x^3 - x^2 + 15x - 10 & 4x^4 + 6x^3 - 4x^2 - 15x - 15 & 2 \\
 & 4x^4 - 12x^3 - 2x^2 + 30x - 20 & \\
\hline
 & 18x^3 - 2x^2 - 45x + 5 &
\end{array}$$

The remainder $= 2x^2(9x - 1) - 5(9x - 1) = (2x^2 - 5)(9x - 1)$.
The factor $2x^2 - 5$ is the H.C.F. required.

LOWEST COMMON MULTIPLE

93. A **common multiple** of two or more integral and rational expressions is an integral and rational expression that is exactly divisible by each of the expressions.

The **lowest common multiple** of two or more integral and rational expressions is an integral and rational expression of

lowest degree and of smallest numerical coefficient **that is exactly divisible by each of the given expressions.**

For brevity, L.C.M. will be used for lowest common multiple.

Find the L.C.M. of $12\,a^2c$, $14\,bc^2$, $36\,ab^2$.

$$12\,a^2c = 2^2 \times 3\,a^2c,$$
$$14\,bc^2 = 2 \times 7\,bc^2,$$
$$36\,ab^2 = 2^2 \times 3^2\,ab^2.$$
$$\therefore \text{ the L.C.M.} = 2^2 \times 3^2 \times 7\,a^2b^2c^2 = 252\,a^2b^2c^2.$$

Hence, **to find the L.C.M. of two or more expressions,**

Resolve each expression into its prime factors.

The product of all the different factors, each factor being taken the greatest number of times it occurs in any of the given expressions, is the lowest common multiple required.

94. When the expressions cannot be readily resolved into their factors, the expressions may be resolved by finding their H.C.F.

Find the L.C.M. of

$$6\,x^3 - 11\,x^2y + 2\,y^3 \text{ and } 9\,x^3 - 22\,xy^2 - 8\,y^3.$$

$6 - 11 + 0 + 2$	$9 + 0 - 22 - 8$	
$6 - 8 - 4$	2	
$-3 + 4 + 2$	$18 + 0 - 44 - 16$	3
$-3 + 4 + 2$	$18 - 33 + 0 + 6$	
	$11\,)\,33 - 44 - 22$	
	$3 - 4 - 2$	$2 - 1$

Hence, $6\,x^3 - 11\,x^2y + 2\,y^3 = (2\,x - y)(3\,x^2 - 4\,xy - 2\,y^2)$,

and $9\,x^3 - 22\,xy^2 - 8\,y^3 = (3\,x + 4\,y)(3\,x^2 - 4\,xy - 2\,y^2)$.

$\therefore$ the L.C.M. $= (2\,x - y)(3\,x + 4\,y)(3\,x^2 - 4\,xy - 2\,y^2)$.

In this example we find the H.C.F. of the given expressions and divide each of them by the H.C.F.

Instead of dividing each expression by the H.C.F., we may divide only one expression, and multiply the quotient by the other expression.

95. *The product of the H.C.F. and the L.C.M. of two expressions is equal to the product of the given expressions.*

Let A and B stand for any two expressions; and let F stand for their H.C.F. and M for their L.C.M.

Let a and b be the quotients when A and B respectively are divided by F. Then,

$$A = aF,$$

and
$$B = bF.$$

Therefore,
$$AB = F \times abF. \qquad [1]$$

Since F stands for the H.C.F. of A and B, F contains *all the common factors* of A and B. Therefore, a and b have no common factor, and abF is the L.C.M. of A and B.

Put M for its equal, abF, in equation [1], and we have

$$AB = FM.$$

96. Since
$$FM = AB,$$

$$M = \frac{AB}{F} = \frac{A}{F} \times B,$$

or
$$M = \frac{AB}{F} = \frac{B}{F} \times A.$$

That is: *The lowest common multiple of two expressions may be found by dividing their product by their highest common factor, or by dividing* either *of them by their highest common factor and multiplying the quotient by the other.*

97. The H.C.F. of three or more expressions is obtained by finding the H.C.F. of two of them; then the H.C.F. of this result and of the third expression; and so on.

For, if A, B, and C stand for three expressions,
and D for the highest common factor of A and B,
and E for the highest common factor of D and C,
then D contains every factor common to A and B,
and E contains every factor common to D and C;
that is, E contains every factor common to A, B, and C.

98. The L.C.M. of three or more expressions may be obtained by finding the L.C.M. of two of them; then the L.C.M. of this result and of the third expression; and so on.

For, if A, B, and C stand for three expressions,
and L for the lowest common multiple of A and B,
and M for the lowest common multiple of L and C,
then L is the expression of lowest degree that is exactly divisible by A and B,
and M is the expression of lowest degree that is exactly divisible by L and C.

That is, M is the expression of lowest degree that is exactly divisible by A, B, and C.

Exercise 6

Find the H.C.F. of:

1. $12x^2 - 17x + 6,\ 9x^2 + 6x - 8.$
2. $x^4 - a^4,\ x^2 + 3ax - 4a^2,\ x^2 - 5ax + 4a^2.$
3. $x^4 - 6x^3 + 13x^2 - 12x + 4,\ x^4 - 4x^3 + 8x^2 - 16x + 16.$
4. $3x^4 - x^3 - 2x^2 + 2x - 8,\ 6x^4 + 13x^3 + 3x^2 + 20x.$
5. $96x^4 + 8x^3 - 2x,\ 32x^3 - 24x^2 - 8x + 3.$
6. $x^4 + 5x^3 - 7x^2 - 9x - 10,\ 2x^4 - 4x^3 + 4x - 8.$
7. $2x^3 - 16x + 6,\ 5x^6 + 15x^5 + 5x + 15.$
8. $2a^4 + 3a^3x - 9a^2x^2,\ 6a^4x - 3ax^4 - 17a^3x^2 + 14a^2x^3.$
9. $2a^5 - 4a^4 + 8a^3 - 12a^2 + 6a,$
   $3a^6 - 3a^5 - 6a^4 + 9a^3 - 3a^2.$
10. $3x^3 - 7x^2y - y^3 + 5xy^2,\ x^2y + 3xy^2 - 3x^3 - y^3,$
    $3x^3 + 5x^2y + xy^2 - y^3.$
11. $36x^7 - 28x^5 + 32x^4 + 8x^3 - 16x^2,$
    $12x^5 - 14x^4 - 20x^3 + 10x^2 + 4x.$

12. $15x^4 + 2x^3 - 75x^2 + 5x + 2,$
$35x^4 + x^3 - 175x^2 + 30x + 1.$

13. $21x^4 - 4x^3 - 15x^2 - 2x,\ 21x^3 - 32x^2 - 54x - 7.$

14. $9x^4y - 22x^2y^3 - 3xy^4 + 10y^5,$
$9x^5y - 6x^4y^2 + x^3y^3 - 25xy^5.$

15. $6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1,$
$4x^4 + 2x^3 - 18x^2 + 3x - 5.$

16. $x^4 - ax^3 - a^2x^2 - a^3x - 2a^4,\ 3x^3 - 7ax^2 + 3a^2x - 2a^3.$

17. $12(x^4 - y^4),\ 10(x^6 - y^6),\ 8(x^4y + xy^4).$

18. $x^4 + xy^3,\ x^3y + y^4,\ x^4 + x^2y^2 + y^4.$

19. $2(x^2y - xy^2),\ 3(x^3y - xy^3),\ 4(x^4y - xy^4),\ 5(x^5y - xy^5).$

Find the L.C.M. of:

20. $x^2 - 3x - 4,\ x^2 - x - 12,\ x^2 + 5x + 4.$

21. $6x^2 - 13x + 6,\ 6x^2 + 5x - 6,\ 9x^2 - 4.$

22. $3x^4 - x^3 - 2x^2 + 2x - 8,\ 6x^3 + 13x^2 + 3x + 20.$

23. $15a^3x^4 + 10a^4x^3 + 4a^5x^2 + 6a^6x - 3a^7,$
$12x^4 + 38ax^3 + 16a^2x^2 - 10a^3x.$

24. $2x^4 + x^3 - 8x^2 - x + 6,\ 4x^4 + 12x^3 - x^2 - 27x - 18,$
$4x^4 + 4x^3 - 17x^2 - 9x + 18.$

25. $x^5 - 2x^4 + x^3,\ 2x^4 - 4x^3 - 4x - 4.$

26. $x^3 - 6x^2 + 11x - 6,\ x^3 - 9x^2 + 26x - 24,$
$x^3 - 8x^2 + 19x - 12.$

27. $4x^3 - x^2y - 3xy^2,\ 3x^3 - 3x^2y + xy^2 - y^3.$

28. $4x^3 - 12x^2 + 9x - 1,\ x^4 - 2x^3 + x^2 - 8x + 8.$

29. $2x^5 - 8x^4 + 12x^3 - 8x^2 + 2x,\ 3x^5 - 6x^3 + 3x.$

30. $x^3 - 6x^2 + 5x + 12,\ x^3 - 5x^2 + 2x + 8,$
$x^3 - 4x^2 + x + 6.$

CHAPTER IV

SYMMETRY

99. Symmetrical Expressions. An algebraic expression that involves two or more letters is *symmetrical with respect to any two of them* if these letters can be interchanged without altering the expression in value or in form.

Thus, $x^2 + a^2x + ab + b^2x$ is symmetrical with respect to a and b; for if a is substituted for b and b for a, the expression becomes

$$x^2 + b^2x + ba + a^2x,$$

which differs from $x^2 + a^2x + ab + b^2x$ only in the order of its terms and the order of their factors.

Again, $x^2 + a^2x + ab + b^2x$ is not symmetrical with respect to a and x; for if a is substituted for x and x for a, the expression becomes

$$a^2 + x^2a + xb + b^2a,$$

which differs in form from $x^2 + a^2x + ab + b^2x$.

In like manner it may be shown that $x^2 + a^2x + ab + b^2x$ is not symmetrical with respect to x and b.

An expression is *symmetrical with respect to three or more of its letters* if it is symmetrical with respect to each and every pair of these letters that can be selected.

Thus, $x^3 + y^3 + z^3 - 3xyz + ab$ is symmetrical with respect to x, y, and z; for it remains the same if x and y are interchanged, or if y and z or x and z are interchanged.

An expression is *completely symmetrical* if it is symmetrical with respect to each and every pair of its letters that can be selected.

Thus, $x^3 + y^3 + z^3 + 3xyz$ is completely symmetrical; for it remains the same if x is interchanged with y, y with z, or z with x; and these three pairs are all the pairs that can be selected from x, y, and z.

100. Cyclo-Symmetrical Expressions. An algebraic expression is *cyclo-symmetrical with respect to certain letters in a given order* when the value and form of the expression is not altered by substituting the second letter for the first, the third for the second, and so on, and the first for the last.

Thus, the expression $ab + bc + cd + da$ is cyclo-symmetrical with respect to the *cycle* $(abcd)$ but is not completely symmetrical with respect to a, b, c, and d.

Every expression that is symmetrical with respect to a set of letters is also cyclo-symmetrical with respect to these letters; but as is seen by the last illustration an expression may be cyclo-symmetrical with respect to a set of letters without being symmetrical with respect to the letters.

101. Principle of Symmetry. An expression which in any one form is symmetrical or cyclo-symmetrical with respect to any set of letters will in every other form be symmetrical or cyclo-symmetrical, as the case may be, with respect to these letters.

Thus, $a^3 + b^3 + c^3 - 3abc$ is symmetrical with respect to a, b, and c. Hence, it is symmetrical with respect to a, b, and c when written in any other form, as $\frac{1}{2}(a + b + c)[(b - c)^2 + (c - a)^2 + (a - b)^2]$.

Again, $(a - b)^3 + (b - c)^3 + (c - a)^3$ is cyclo-symmetrical with respect to (a, b, c), but not completely symmetrical. Hence, it remains cyclo-symmetrical with respect to (a, b, c), but not completely symmetrical when written in any other form, as $3(a - b)(b - c)(c - a)$.

102. Σ Notation. A symmetrical expression is often written by writing each *type-term* once, preceded by the Greek letter Σ, where Σ stands for the words *the sum of all the terms of the same type as.*

Thus,
$$\Sigma a = a + b + c + d + \cdots$$
$$\Sigma ab = ab + bc + cd + \cdots + bc + bd + \cdots + cd + \cdots$$

If the three letters, a, b, c, are involved,
$$\Sigma a^2b = a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2.$$

Exercise 7

1. For a, b, c, write the following in full:

$$\Sigma(a-b)^2; \qquad \Sigma a(b-c); \qquad \Sigma a^2b^2c;$$
$$\Sigma[(a+c)^2-b^2]; \quad \Sigma a(b+c)^2; \quad \Sigma(a+b)(c-a)(c-b).$$

2. For a, b, c, d, write the following in full:

$$\Sigma abc; \qquad \Sigma a^2b; \qquad \Sigma a^2bc; \qquad \Sigma(a-b);$$
$$\Sigma a^2(a-b); \qquad \Sigma a^2b^3c; \qquad \Sigma(a+b-c).$$

Show that the following expressions are symmetrical:

3. $(x+a)(a+b)(b+x)+abx$, with respect to a and b.

4. $(a+b)^2+(a-b)^2$, with respect to a and b, and also with respect to a and $-b$.

5. $a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2$, with respect to a, b, c.

6. $(ac+bd)^2+(bc-ad)^2$, with respect to a^2 and b^2, and also with respect to c^2 and d^2.

Select the letters with respect to which the following expressions are symmetrical:

7. $(a^2-c^2)^2+4\,b^2(a+c)^2+(2\,ac-2\,b^2)^2$.

8. $x^6-y^6+z^6-3\,(x^2-y^2)(y^2-z^2)(z^2+x^2)$.

9. $a^2b^2+b^2c^2+c^2a^2-2\,abc\,(a+b-c)$.

10. $(a+b)^4+(a-c)^4+(b+c)^4+(a+c)^4$.

11. Show that $(a-b)^2(b-c)(c-d)(a-c)(b-d)^2(a-d)^2$ is not symmetrical with respect to a, b, c, and d. With respect to which of its letters is the expression symmetrical?

103. Rule of Symmetry. The applications of symmetry and of cyclo-symmetry are numerous. From the definitions given in §§ 99 and 100, we have the following:

The sum, the difference, the product, or the quotient of two symmetrical or cyclo-symmetrical expressions is also a symmetrical or a cyclo-symmetrical expression.

104. In reducing a symmetrical expression from one form to another, advantage may be taken of the principles of symmetry; for it is necessary to calculate only the *type-terms*. The other terms may be written at once from these.

(1) Simplify $(a+b+c+d+e+\cdots)^2$.

The expression is symmetrical with respect to $a, b, c, \cdots$; hence, the expansion also is symmetrical, and as it is a product of *two* factors, it can contain only the squares $a^2, b^2, c^2, \cdots$, and the products $ab, ac, ad, \cdots, bc, bd, \cdots$; so that the *type-terms* are a^2 and ab.

Now $(a+b)^2 = a^2 + 2\,ab + b^2$; and the addition of terms involving $c, d, e, \cdots$ does not alter the terms $a^2 + 2\,ab$, but merely gives additional terms of the same type. Hence, from symmetry,

$$\begin{aligned}(a+b+c+d+e+\cdots)^2 = a^2 &+ 2\,ab + 2\,ac + 2\,ad + 2\,ae + \cdots\\ &+ b^2 + 2\,bc + 2\,bd + 2\,be + \cdots\\ &+ c^2 + 2\,cd + 2\,ce + \cdots\\ &+ d^2 + 2\,de + \cdots\\ &+ e^2 + \cdots\end{aligned}$$

This equation may be compactly written,

$$(\Sigma a)^2 = \Sigma a^2 + 2\,\Sigma ab.$$

(2) Simplify $(a+b)^3$.

The expression is of *three* dimensions, and is symmetrical with respect to a and b.

The type-terms are a^3, a^2b.

$\therefore (a+b)^3 = a^3 + b^3 + n(a^2b + b^2a)$, where n is *numerical*.

To find the value of n, substitute for a and b any convenient values that will not reduce either side of the equation to 0, as for instance put $a = b = 1$.

Then, $$(1+1)^3 = 1^3 + 1^3 + n(1^2 \times 1 + 1^2 \times 1).$$

Whence, $$n = 3.$$

$$\therefore (a+b)^3 = \Sigma a^3 + 3\,\Sigma a^2b.$$

(3) Simplify $(x+y+z)^3$.

The expression is of three dimensions and is symmetrical with respect to x, y, and z. We have

$$(x+y+z)^3 = [(x+y)+z]^3 = (x+y)^3 + \cdots = x^3 + 3\,x^2y + \cdots$$

The type-terms are x^3 and $3\,x^2y$, and the only other possible type-term is xyz.

Now, since the expression contains $3x^2y$, it must also contain $3x^2z$; that is, it must contain $3x^2(y+z)$.

Hence,
$$\begin{aligned}(x+y+z)^3 = x^3 &+ 3x^2(y+z)\\ + y^3 &+ 3y^2(z+x)\\ + z^3 &+ 3z^2(x+y)\\ &+ n(xyz),\end{aligned}$$

where n is numerical, and is found to be equal to 6 by putting $x=y=z=1$ in the last equation.

$$\therefore (x+y+z)^3 = \Sigma x^3 + 3\,\Sigma x^2y + 6\,xyz.$$

(4) Simplify $(a+b+c+\cdots)^3$.

The type-terms are a^3, a^2b, abc.

Simplifying $(a+b+c)^3$, we obtain $a^3 + 3\,a^2b + 6\,abc + \cdots$

Hence, by symmetry, we have

$$(\Sigma a)^3 = \Sigma a^3 + 3\,\Sigma a^2b + 6\,\Sigma abc.$$

(5) Simplify

$$(x+y+z)^3 + (x-y-z)^3 + (y-z-x)^3 + (z-x-y)^3.$$

The expression is symmetrical with respect to x, y, and z.

The type-terms are x^3, $3\,x^2y$, $6\,xyz$.

x^3 occurs in each of the first two cubes, and $-x^3$ in each of the second two cubes. Therefore, in the result there are no terms of the type x^3.

$3\,x^2y$ occurs in the first and third cubes, and $-3\,x^2y$ in the second and fourth. Therefore, in the result there are no terms of the type $3\,x^2y$.

$6\,xyz$ occurs in each of the four cubes.

Therefore, the given expression simplifies to $24\,xyz$.

Exercise 8

Simplify:

1. $(a+b+c)^2 + (a+b-c)^2 + (b+c-a)^2 + (c+a-b)^2$.
2. $(a+b+c)^2 - a(b+c-a) - b(a+c-b) - c(a+b-c)$.
3. $(x+y+z+n)^2 + (x-y-z+n)^2$
   $+ (x-y+z-n)^2 + (x+y-z-n)^2$.
4. $(x-2\,y-3\,z)^2 + (y-2\,z-3\,x)^2 + (z-2\,x-3\,y)^2$.
5. $a(b+c)(b^2+c^2-a^2) + b(c+a)(c^2+a^2-b^2)$
   $+ c(a+b)(a^2+b^2-c^2)$.
6. $(ab+bc+ca)^2 - 2\,abc(a+b+c)$.

Prove that:

7. $(a+b+c)^4+(b+c-a)^4+(c+a-b)^4+(a+b-c)^4$
$=4(a^4+b^4+c^4)+24(a^2b^2+b^2c^2+c^2a^2).$

8. $(a+b+c)^4=\Sigma a^4+4\,\Sigma a^3b+6\,\Sigma a^2b^2+12\,\Sigma a^2bc.$

9. $(\Sigma a)^4=\Sigma a^4+4\,\Sigma a^3b+6\,\Sigma a^2b^2+12\,\Sigma a^2bc+24\,\Sigma abcd.$

10. $(a-b)^2(b-c)^2+(b-c)^2(c-a)^2+(c-a)^2(a-b)^2$
$=(a^2+b^2+c^2-ab-ac-bc)^2.$

11. $(ar^2+2\,brs+cs^2)(ax^2+2\,bxy+cy^2)$
$-[arx+b(ry+sx)+csy]^2=(ac-b^2)(ry-sx)^2.$

12. $(a^2+ab+b^2)(c^2+cd+d^2)=(ac+ad+bd)^2$
$+(ac+ad+bd)(bc-ad)+(bc-ad)^2.$

105. Factoring. The principles of symmetry can be used in resolving expressions into factors.

(1) Find the factors of

$$(a+b+c)(ab+bc+ca)-(a+b)(b+c)(c+a).$$

The expression is symmetrical with respect to a, b, and c.

If there is a *monomial* factor, a must be one. If we put 0 for a, the expression vanishes. Hence, a is a factor, § 87, p. 44, and by symmetry b and c are also factors. Therefore, abc is a factor.

There can be no other *literal* factor, for the given expression is of only three dimensions and abc is of three dimensions.

There may be a *numerical* factor, however. Let m be a numerical factor of the given expression.

Then $(a+b+c)(ab+bc+ca)-(a+b)(b+c)(c+a)=mabc.$

To find m, put $a=b=c=1$ in this equation, and $m=1$.

Therefore, the given expression is equal to abc.

(2) Find the factors of

$$a^3(b-c)+b^3(c-a)+c^3(a-b).$$

If we put $a=0$, the expression does not vanish. Hence, a is not a factor, and by symmetry neither b nor c is a factor.

If we put $a=b$ in the expression, the expression vanishes. Hence, $a-b$ is a factor, § 87, p. 44, and by symmetry $b-c$ and $c-a$ are factors.

Now the given expression is of four dimensions. Hence, in addition to the three factors already found there must be one other factor of one dimension; and as this factor must be symmetrical with respect to a, b, and c, it must be $a+b+c$.

There may be a numerical factor.

Let m be a numerical factor of the given expression. Then,

$$a^3(b-c)+b^3(c-a)+c^3(a-b)=m(a-b)(b-c)(c-a)(a+b+c).$$

To find m, put $\quad a=0,\ b=1,\ c=2.$

Then, $\quad m=-1.$

Hence, $\quad$ the given expression $=-(a-b)(b-c)(c-a)(a+b+c).$

(3) Prove that $a^3+b^3+c^3+3(a+b)(b+c)(c+a)$ is exactly divisible by $a+b+c$, and find all the factors.

Let $a+b+c=0$, or $a=-(b+c)$, and substitute this value of a.

Then the given expression becomes $-(b+c)^3+b^3+c^3+3bc(b+c)$ or $-(b+c)^3+(b+c)^3$, or 0.

Hence, $a+b+c$ is a factor.

If we put $a=0$, the expression does not vanish. Hence, a is not a factor, and by symmetry b and c are not factors.

Since $a+b+c$, the factor already obtained, is of *one* dimension, the other factor must be of *two* dimensions, and since it must be symmetrical with respect to a, b, and c, it must be of the form

$$m(a^2+b^2+c^2)+n(ab+bc+ca),$$

in which m and n are independent of each other, and of a, b, and c.

To determine the values of m and n, put $c=0$ in the equation

$$a^3+b^3+c^3+3(a+b)(b+c)(c+a)$$
$$=(a+b+c)[m(a^2+b^2+c^2)+n(ab+bc+ca)].$$

Then, $\quad a^3+b^3+3ab(a+b)=(a+b)[m(a^2+b^2)+nab].$

But $\quad a^3+b^3+3ab(a+b)=(a+b)^3.$

Therefore, $\quad (a+b)^3=(a+b)[m(a^2+b^2)+nab].$

Therefore, $\quad (a+b)^2=m(a^2+b^2)+nab.$

That is, $\quad (a^2+b^2)+2ab=m(a^2+b^2)+nab.$

Now, this equation is true for *all* values of a and b.

Therefore, $\quad m=1$, and $n=2$.

$$\therefore\ a^3+b^3+c^3+3(a+b)(b+c)(c+a)$$
$$=(a+b+c)[a^2+b^2+c^2+2(ab+bc+ca)]$$
$$=(a+b+c)(a+b+c)^2$$
$$=(a+b+c)^3.$$

(4) Show that $x^n + 1$ is a factor of $x^{3n} + 2x^{2n} + 3x^n + 2$.

Let $x^n + 1 = 0$, or $x^n = -1$, and substitute.
Then, $x^{3n} + 2x^{2n} + 3x^n + 2 = -1 + 2 - 3 + 2 = 0$.
Therefore, $x^n + 1$ is a factor of the given expression.

(5) Show that $a^2 + b^2$ is a factor of $2a^4 + a^3b + 2a^2b^2 + ab^3$

Let $a^2 + b^2 = 0$, or $a^2 = -b^2$, and substitute.
Then, $2a^4 + a^3b + 2a^2b^2 + ab^3 = 2b^4 - ab^3 - 2b^4 + ab^3 = 0$.
Therefore, $a^2 + b^2$ is a factor of the given expression.

Exercise 9

Resolve into factors:

1. $(x + y + z)^3 - (x^3 + y^3 + z^3)$.
2. $bc(b - c) - ca(a - c) - ab(b - a)$.
3. $(a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3$.
4. $x(y + z)^2 + y(z + x)^2 + z(x + y)^2 - 4xyz$.
5. $(a + b)^3 - (b + c)^3 + (c - a)^3$.
6. $a(b - c)^3 + b(c - a)^3 + c(a - b)^3$.
7. $(a + b + c)(ab + bc + ca) - abc$.
8. $a^3(c - b^2) + b^3(a - c^2) + c^3(b - a^2) + abc(abc - 1)$.
9. $a^2(b + c) + b^2(c + a) + c^2(a + b) + 2abc$.
10. $x^4y^2 + x^2y^4 + x^4z^2 + x^2z^4 + y^4z^2 + y^2z^4 + 2x^2y^2z^2$.
11. $(a - b)^5 + (b - c)^5 + (c - a)^5$.
12. $ab(a + b) + bc(b + c) + ca(c + a) + (a^3 + b^3 + c^3)$.
13. $a^4(c - b^3) + b^4(a - c^3) + c^4(b - a^3) + abc(a^2b^2c^2 - 1)$.
14. $x^4(y^2 - z^2) + y^4(z^2 - x^2) + z^4(x^2 - y^2)$.
15. $x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2$.
16. $(a + b)^3 + (b + c)^3 + (c + a)^3$
    $+ 3(a + 2b + c)(b + 2c + a)(c + 2a + b)$.
17. $a^4(b - c) + b^4(c - a) + c^4(a - b)$.
18. Show that $a^5 + a^2b^2 - ab^2 - b^3$ has $a^2 - b$ for a factor.

CHAPTER V

FRACTIONS

106. An algebraic expression is **integral** when it consists of a number of terms connected by + and − signs, and each term is the product of a coefficient into positive integral powers of the letters involved.

In an integral algebraic expression the *coefficients* may be fractional.

Thus, $x^3 - \frac{1}{3} ax^2 + \frac{2}{3} a$ is an *integral* algebraic expression.

107. An **algebraic fraction** is the indicated quotient of two algebraic expressions, and is generally written in the form $\frac{a}{b}$.

The dividend, a, is called the **numerator**; the divisor, b, the **denominator.**

The numerator and denominator are called the **terms** of the fraction.

108. Since the quotient is unchanged if the dividend and divisor are both multiplied (or divided) by the same factor, the value of a fraction is unchanged if the numerator and denominator are multiplied (or divided) by the same factor.

109. To reduce a fraction to lower terms,

Divide the numerator and the denominator by any common factor.

A fraction is expressed in its **lowest terms** when both numerator and denominator are divided by their H.C.F.

(1) Reduce to lowest terms $\dfrac{6x^2 - 5x - 6}{8x^2 - 2x - 15}$.

By § 83, p. 42, $\dfrac{6x^2 - 5x - 6}{8x^2 - 2x - 15} = \dfrac{(2x - 3)(3x + 2)}{(2x - 3)(4x + 5)} = \dfrac{3x + 2}{4x + 5}$.

(2) Reduce to lowest terms $\frac{a^3 - 7a^2 + 16a - 12}{3a^3 - 14a^2 + 16a}$.

Since no common factor can be determined by inspection, it is necessary to find the H.C.F. of the numerator and denominator by the method of division.

We find the H.C.F. to be $a - 2$.

Now, if $a^3 - 7a^2 + 16a - 12$ is divided by $a - 2$, the result is $a^2 - 5a + 6$; and if $3a^3 - 14a^2 + 16a$ is divided by $a - 2$, the result is $3a^2 - 8a$.

$$\therefore \frac{a^3 - 7a^2 + 16a - 12}{3a^3 - 14a^2 + 16a} = \frac{a^2 - 5a + 6}{3a^2 - 8a}.$$

110. Mixed Expressions. If the degree of the numerator of a fraction equals or exceeds that of the denominator, the fraction may be changed to the form of an integral or a mixed expression *by dividing the numerator by the denominator.*

The quotient is the integral expression; the remainder (if any) is the numerator, and the divisor the denominator, of the fractional expression.

To reduce a mixed expression to a fractional form,

Multiply the integral expression by the denominator, to the product add the numerator, and under the result write the denominator.

The dividing line has the force of a vinculum or parenthesis affecting the numerator; therefore, if a *minus sign* precedes the dividing line, and this line is removed, the sign of *every term* of the numerator must be *changed.*

Thus, $$n - \frac{a-b}{c} = \frac{cn - (a-b)}{c} = \frac{cn - a + b}{c}.$$

111. To reduce fractions to equivalent fractions having the lowest common denominator,

Find the L.C.M. of the denominators.

Divide the L.C.M. by the denominator of each fraction.

Multiply the first numerator by the first quotient, the second numerator by the second quotient, and so on.

The products are the numerators of the equivalent fractions.

The L.C.M. of the given denominators is the denominator of each of the equivalent fractions.

Thus, $\frac{3x}{4a^2}, \frac{2y}{3a}, \frac{5}{6a^3}$ are equal to $\frac{9ax}{12a^3}, \frac{8a^2y}{12a^3}, \frac{10}{12a^3}$, respectively.

The multipliers $3a$, $4a^2$, and 2 are obtained by dividing $12a^3$, the L.C.M. of the denominators, by the respective denominators of the given fractions.

112. To add fractions,

Reduce the fractions to equivalent fractions having the lowest common denominator.

Add the numerators of the equivalent fractions.

Write the result over the lowest common denominator.

To subtract one fraction from another we proceed as in addition, except that we *subtract* the numerator of the subtrahend from that of the minuend.

(1) Simplify $\frac{3a-4b}{7}-\frac{2a-b+c}{3}+\frac{13a-4c}{12}$.

The L.C.D. is 84.

The multipliers are 12, 28, and 7 respectively.

$$\begin{array}{rl} 36a - 48b & = \text{1st numerator,} \\ -56a + 28b - 28c & = \text{2d numerator,} \\ 91a - 28c & = \text{3d numerator.} \\ \hline 71a - 20b - 56c & = \text{sum of numerators.} \end{array}$$

$$\therefore \frac{3a-4b}{7}-\frac{2a-b+c}{3}+\frac{13a-4c}{12}=\frac{71a-20b-56c}{84}.$$

Since the *minus sign* precedes the second fraction, the signs of all the terms of the numerator of this fraction are changed after being multiplied by 28.

(2) Simplify $\frac{y^2}{x^2-y^2}-\frac{x-y}{x+y}+\frac{2xy}{x^2+y^2}+1$.

The L.C.D. is $(x+y)(x-y)(x^2+y^2)$.

The multipliers are

x^2+y^2, $(x-y)(x^2+y^2)$, $(x+y)(x-y)$, $(x+y)(x-y)(x^2+y^2)$

respectively.

$$\begin{array}{rcl}
x^2y^2 \qquad\qquad + y^4 &=& \text{1st numerator,}\\
-x^4+2x^3y-2x^2y^2+2xy^3-y^4 &=& \text{2d numerator,}\\
2x^3y \qquad -2xy^3 \quad &=& \text{3d numerator,}\\
x^4 \qquad\qquad\qquad -y^4 &=& \text{4th numerator.}\\
\hline
4x^3y-\ x^2y^2 \qquad -y^4 &=& \text{sum of numerators.}
\end{array}$$

$$\therefore \text{ sum of fractions} = \frac{4x^3y-x^2y^2-y^4}{x^4-y^4}.$$

113. Since $\frac{ab}{b}=a$, and $\frac{-ab}{-b}=a$, it is evident that if the signs of both numerator and denominator are changed, the value of the fraction is not altered.

Since changing the sign before the fraction is equivalent to changing the sign before every term of the numerator or denominator, therefore *the sign before every term of the numerator or denominator may be changed, provided the sign before the fraction is changed.*

Since, also, the product of $+a$ multiplied by $+b$ is ab, and the product of $-a$ multiplied by $-b$ is ab, the signs of *two factors*, or of *any even number of factors*, of the numerator or denominator of a fraction may be changed without altering the value of the fraction.

By the application of these principles, fractions may often be changed to a form more convenient for addition or subtraction.

Simplify $\frac{2}{x}-\frac{3}{2x-1}+\frac{2x-3}{1-4x^2}$.

Change the signs before the terms of the denominator of the third fraction, and change the sign before the fraction.

The result is $\frac{2}{x}-\frac{3}{2x-1}-\frac{2x-3}{4x^2-1}$, in which the several denominators are written in similar form.

The L.C.D. is $x(2x-1)(2x+1)$.

$$\begin{array}{rl} 8x^2 \qquad\quad -2 &= \text{1st numerator,} \\ -6x^2 - 3x \qquad &= \text{2d numerator,} \\ \underline{-2x^2 + 3x \qquad} &= \text{3d numerator.} \\ -2 &= \text{sum of numerators.} \end{array}$$

$$\therefore \text{sum of the fractions} = \frac{-2}{x(2x-1)(2x+1)}.$$

114. Multiplication of Fractions. Let it be required to find the product of the two fractions $\frac{a}{b}$ and $\frac{c}{d}$.

If we multiply the dividend a by c, we multiply the quotient $\frac{a}{b}$ by c; if we multiply the divisor b by d, we divide the quotient $\frac{a}{b}$ by d. Hence, the product of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{ac}{bd}$.

Therefore, **to find the product of two fractions,**

Find the product of the numerators for the numerator of the product, and the product of the denominators for the denominator of the product.

115. Division of Fractions. Multiplying by the reciprocal of a number is equivalent to dividing by the number.

The reciprocal of a fraction is the fraction with its terms interchanged. Therefore, **to divide by a fraction,**

Interchange the terms of the fraction and multiply by the resulting fraction.

If the divisor is an integral expression, it may be changed to the fractional form.

116. A **complex fraction** is a fraction which has a fraction in the numerator, or in the denominator, or in both terms.

To simplify a complex fraction,

Divide the numerator by the denominator.

Or we may multiply both terms of the fraction by the L.C.D. of the fractions contained in the numerator and denominator.

Exercise 10

Reduce to lowest terms:

1. $\dfrac{42a^3 - 30a^2x}{35ax^2 - 25x^3}$.

2. $\dfrac{2x^3 + 5x^2 - 12x}{7x^3 + 25x^2 - 12x}$.

3. $\dfrac{6a^2c^2 - 2a^4 + 18c^2 - 6a^2}{4a^4 + 2a^2c^2 + 12a^2 + 6c^2}$.

4. $\dfrac{x^4 + (2b^2 - a^2)x^2 + b^4}{x^4 + 2ax^3 + a^2x^2 - b^4}$.

5. $\dfrac{6x^5 - 9x^4 + 11x^3 + 6x^2 - 10x}{4x^6 + 10x^5 + 10x^4 + 4x^3 + 60x^2}$.

Simplify:

6. $\dfrac{3x - 2y}{3} - \dfrac{4y + 2x}{5} + \dfrac{22y - 9x}{15}$.

7. $\dfrac{2}{3a} - \dfrac{1}{2b} - \dfrac{2a + 3}{6a^2} + \dfrac{1}{2x^2} + \dfrac{3a - 2b}{6ab}$.

8. $\dfrac{3}{x - a} + \dfrac{4a}{(x - a)^2} - \dfrac{5a^2}{(x - a)^3}$.

9. $\dfrac{a + b}{(b - c)(c - a)} + \dfrac{b + c}{(c - a)(a - b)} - \dfrac{a - c}{(a - b)(b - c)}$.

10. $\dfrac{1}{a(a - b)(a - c)} + \dfrac{1}{b(b - c)(b - a)} + \dfrac{1}{c(c - a)(c - b)}$.

11. $\dfrac{16x^2 - 17x + 12}{12x^2 - 25x + 12} + \dfrac{27x^2 + 18x - 24}{12x^2 + 7x - 12} + \dfrac{25x^2 - 25x + 6}{20x^2 - 23x + 6}$.

12. $\dfrac{2a^8x^7}{3b^3} \times \dfrac{5a^4b^5}{4c^4x^6} \times \dfrac{15b^2c^2}{4a^9x} \div \dfrac{25a^4x}{18ab^2c^3}$.

13. $\left(\dfrac{x^4 - y^4}{x^2 - y^2} \div \dfrac{x + y}{x^2 - xy}\right) \div \left(\dfrac{x^2 + y^2}{x - y} \div \dfrac{x + y}{xy - y^2}\right)$.

14. $\left(\dfrac{a^2 + b^2}{b} - a\right)\left(\dfrac{a^2 - b^2}{a^3 + b^3}\right) \div \left(\dfrac{1}{b} - \dfrac{1}{a}\right)$.

15. $\dfrac{x^2-7x+12}{x^2+5x+6}\times\dfrac{x^2+x-2}{x^2-5x+4}\times\dfrac{2x^2+5x-3}{3x^2-7x-6}$.

16. $\dfrac{6a^2-a-2}{8a^2-2a-3}\times\dfrac{8a^2-10a+3}{12a^2+a-6}\times\dfrac{12a^2+17a+6}{6a^2+a-2}$.

17. $\dfrac{\dfrac{2x+y}{y}-\dfrac{y}{2x+y}}{\dfrac{x}{x+y}-\dfrac{x+y}{x}}$.

19. $\dfrac{\left(a^2+\dfrac{b^4}{a^2-b^2}\right)(a^2+b^2)}{\dfrac{a}{a+b}+\dfrac{b}{a-b}}$.

18. $\dfrac{\dfrac{1+x}{1+x^2}-\dfrac{1+x^2}{1+x^3}}{\dfrac{1+x^2}{1+x^3}-\dfrac{1+x^3}{1+x^4}}$.

20. $\dfrac{\dfrac{64a^3-96a^2x+36ax^2}{36a^2-729x^2}}{\dfrac{48a^2-27x^2}{8a^2-72ax+162x^2}}$.

21. $\dfrac{x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5}{x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5}\div\dfrac{x^2-2xy+y^2}{x^2+2xy+y^2}$.

22. $\dfrac{\dfrac{2(1-x)}{1+x}+\dfrac{(1-x)^2}{(1+x)^2}+1}{\dfrac{2(1+x)}{1-x}+\left(\dfrac{1+x}{1-x}\right)^2+1}$.

23. $\dfrac{\left(\dfrac{x-a}{x+a}\right)^2+\left(\dfrac{x+a}{x-a}\right)^2-2}{\left(\dfrac{x-a}{x+a}\right)^2+\left(\dfrac{x+a}{x-a}\right)^2+2}$.

24. $\dfrac{2}{x-y}+\dfrac{2}{y-z}+\dfrac{2}{z-x}+\dfrac{(x-y)^2+(y-z)^2+(z-x)^2}{(x-y)(y-z)(z-x)}$.

25. $\dfrac{x+1}{2x-1}-\dfrac{x-1}{2x+1}-\dfrac{1-3x}{x(1-2x)}+\dfrac{x}{x(4x^2-1)}+\dfrac{1}{x(16x^4-1)}$.

26. $\dfrac{\dfrac{1}{a-y}-\dfrac{1}{a-x}+\dfrac{y}{(a-y)^2}-\dfrac{x}{(a-x)^2}}{\dfrac{1}{(a-y)(a-x)^2}-\dfrac{1}{(a-x)(a-y)^2}}$.

CHAPTER VI

SIMPLE EQUATIONS

117. Two different expressions that involve the same symbols will generally have different values for different assumed values of the several symbols; for certain values of the symbols involved the two expressions may have the *same* value.

118. An **equation** is a statement that two expressions have the same value; that is, a statement that two expressions represent the same number.

Every equation consists of two expressions connected by the sign of equality; the two expressions are called the **sides** or **members** of the equation.

An equation will, in general, not hold true for all values of the symbols involved; it will hold true for only those values that give to the two members the same value.

Thus, the equation

$$4x^2 - 3x + 5 = 3x^2 + 4x - 5$$

holds true when for x we put 2, since each member then has the value 15; also when for x we put 5, since each member then has the value 90. If we give to x any other value, the two members will be found to have different values, and the equation will not hold true.

119. An **equation of condition** is an equation that holds true for only *certain particular values* of the symbols involved.

An **identical equation**, or an **identity**, is an equation that holds true for *all values* of the symbols involved.

The two members of an identical equation are **identical expressions**.

In identical equations it is customary to use the sign $\equiv$, called the **sign of identity**, instead of the sign of equality.

Thus, the two expressions $(x+y)^2$ and $x^2+2xy+y^2$ have the same value for all values of x and y, and we accordingly write the identity,

$$(x+y)^2 \equiv x^2+2xy+y^2.$$

This is read $(x+y)^2$ is identically equal to $x^2+2xy+y^2$;
or $(x+y)^2$ is identical with $x^2+2xy+y^2$.

Wherever the term *equation* is used, it is to be understood that an equation of condition is meant, unless the contrary is expressly stated.

In any particular problem we have two kinds of numbers.

1. Numbers that are either given, or supposed to be given, in the problem under consideration. Such numbers are called **known numbers**; if given, they are generally represented by figures; if only supposed to be given, by the first letters of the alphabet.

2. Numbers that are not given in the problem under consideration, but are to be found from certain given relations to the given numbers. Such numbers are called **unknown numbers**, and are generally represented by the last letters of the alphabet.

The relations between the known and unknown numbers are generally expressed by means of equations.

120. **Simultaneous equations** are equations in which the corresponding unknowns have the same values.

In order to find all the unknown numbers in a system of simultaneous equations, we must have as many equations as there are unknown numbers.

121. To **solve** an equation, or a system of simultaneous equations, is to find the unknown numbers involved.

122. The **degree** of an equation is the sum of the exponents of the several unknown numbers in that term in which the sum of the exponents is greatest.

If the equation involves but one unknown number, the degree is the same as the exponent of the highest power of the unknown number involved in the equation.

Equations of the first, second, third, and fourth degrees are called respectively *simple equations, quadratic equations, cubic equations,* and *biquadratic equations.*

123. **Literal equations** are equations in which some or all of the given numbers are represented by letters.

124. An equation that involves but one unknown number, represented for example by x, will hold true for those values of x which give to the two members the same value (§ 118), and for no other values of x. The values of x for which the equation holds true are called the **roots** of the equation.

Thus, the roots of the equation $4x^2 - 3x + 5 = 3x^2 + 4x - 5$ are 2 and 5.

To **solve** an equation that involves one unknown number is, therefore, to find the roots of the equation.

125. The various methods of solving equations are based mainly upon the following general principle:

If similar operations are performed upon equal numbers, the results are equal numbers.

Thus, the two members of a given equation are equal numbers. If the two members are increased by, diminished by, multiplied by, or divided by equal numbers, the results are equal numbers. Similarly, if the two members are raised to like powers, or if like roots of the two members are taken, the results are equal numbers.

126. *Any term may be transposed from one side of an equation to the other, provided its sign is changed.*

Suppose	$x + a = b.$	Suppose	$x - a = b.$
Now,	$a = a.$	Now,	$a = a.$
Subtract,	$x = b - a.$	Add,	$x = a + b.$

To transpose a negative number we add that number to both sides of the equation; to transpose a positive number we subtract that number from both sides.

127. The signs of all the terms on each side of an equation may be changed; for this is in effect transposing every term.

128. To solve a simple equation with one unknown number,

Transpose all the terms involving the unknown number to the left side, and all the other terms to the right side: combine the like terms, and divide both sides by the coefficient of the unknown number.

To *verify* the result, substitute the value of the unknown number in the original equation.

Solve $(x-2)(x+4)=(x+1)(x+2)$.

Expand, $$x^2+2x-8=x^2+3x+2,$$

or $$2x-8=3x+2.$$

Transpose, $$2x-3x=2+8.$$

Combine, $$-x=10.$$

$$\therefore x=-10.$$

129. To clear an equation of fractions,

Multiply each term by the L.C.M. of the denominators.

If a fraction is preceded by a *minus sign*, the sign of every term of the numerator must be *changed* when the denominator is removed.

(1) Solve $\frac{x}{3}-\frac{x-1}{11}=x-9$.

Multiply by 33, the L.C.M. of the denominators.

Then, $$11x-3x+3=33x-297.$$

Transpose and combine, $$-25x=-300.$$

$$\therefore x=12.$$

Since the minus sign precedes the second fraction, in removing the denominator, the + (understood) before x, the first term of the numerator, is changed to −; and the − before 1, the second term of the numerator, is changed to +.

If the denominators contain both simple and compound expressions, it is generally best to remove the simple expressions first, and then each compound expression in turn.

(2) Solve $\frac{8x+5}{14}+\frac{7x-3}{6x+2}=\frac{4x+6}{7}$.

Multiply by 14, $8x+5+\frac{49x-21}{3x+1}=8x+12$.

Transpose and combine, $\frac{49x-21}{3x+1}=7$.

Multiply by $3x+1$, $49x-21=21x+7$.

Transpose and combine, $28x=28$.

$\therefore x=1$.

Exercise 11

Solve:

1. $8(10-x)=5(x+3)$.
2. $2x-3(2x-3)=1-4(x-2)$.
3. $(x-5)(x+6)=(x-1)(x-2)$.
4. $(2x+3)(3x-2)=x^2+x(5x+3)$.
5. $(x-3)(x+5)=(x+1)(2x-3)-x^2$.
6. $(x+4)(x-2)=(x+3)(3x+4)-(2x+1)(x-6)$.
7. $(x-3)(2x+5)=x(x+4)+(x+1)(x+3)$.
8. $(x+2)^2+3x=(x-2)^2+5(16-x)$.
9. $(x-3)^2+(x-4)^2=(x-2)^2+(x+3)^2$.
10. $\frac{3x}{5}-\frac{x}{6}=\frac{26}{15}$.
11. $\frac{x-2}{3x-5}=\frac{6}{19}$.
12. $\frac{3x-5}{2x+10}=\frac{2}{3}$.
13. $\frac{3(5x-3)}{2(4x+3)}=\frac{6}{5}$.
14. $\frac{5x-6}{5}-\frac{3x}{4}=\frac{x-9}{10}$.
15. $\frac{12-3x}{4}-\frac{3x-11}{3}=1$.
16. $\frac{4x+17}{x+3}+\frac{3x-10}{x-4}=7$.
17. $\frac{x-3}{2x+1}+\frac{2x-1}{4x-3}=1$.

18. $\dfrac{4x+3}{3x+4}-\dfrac{3x-4}{4x-3}=\dfrac{7}{12}$.

19. $\dfrac{6x+7}{3}-\dfrac{3}{x+2}=2x+\dfrac{1}{2}$.

20. $\dfrac{2x+1}{a+1}+\dfrac{2x}{a}=5$.

21. $\dfrac{ax-b}{c}-\dfrac{bx+c}{a}=abc$.

22. $\dfrac{x+a}{3(x+b)}+\dfrac{x+b}{2(x+a)}=\dfrac{5}{6}$.

23. $\dfrac{x-2a}{x+3a}-\dfrac{13a^2-2x^2}{x^2-9a^2}=3$.

24. $\dfrac{a}{x}+\dfrac{x}{a}+\dfrac{a(x-a)}{x(x+a)}-\dfrac{x(x+a)}{a(x-a)}=\dfrac{ax}{a^2-x^2}-2$.

130. Problems. In the statement of problems it is to be remembered that the *unit* of the quantity sought is always given, and it is only the *number* of such units that is to be found. We have nothing to do with the quantities themselves; it is only *numbers* with which we have to deal.

Thus, x must never be put for a distance, time, weight, etc., but for a *number* of miles, days, pounds, etc.

(1) A and B had equal sums of money; B gave A \$5, and then 3 times A's money was equal to 11 times B's money. What had each at first?

Let $x=$ the number of dollars each had.
Then $x+5=$ the number of dollars A had after receiving \$5,
and $x-5=$ the number of dollars B had after giving A \$5.

$$\therefore 3(x+5)=11(x-5),$$
$$3x+15=11x-55,$$
$$-8x=-70,$$
$$x=8\tfrac{3}{4}.$$

Therefore, each had at first \$8.75.

(2) A can do a piece of work in 5 days, and B can do it in 4 days. How long will it take A and B to do the work together?

Let x = the number of days it will take A and B together.

Then $\frac{1}{x}$ = the part they can do together in one day.

Now, $\frac{1}{5}$ = the part A can do in one day,

and $\frac{1}{4}$ = the part B can do in one day.

$\therefore \frac{1}{5} + \frac{1}{4}$ = the part A and B can do together in one day.

$$\therefore \frac{1}{5} + \frac{1}{4} = \frac{1}{x},$$

$$4x + 5x = 20,$$

$$9x = 20,$$

$$x = 2\tfrac{2}{9}.$$

Therefore, they can do the work together in $2\frac{2}{9}$ days.

Exercise 12

1. The difference between two numbers is 3; and three times the greater number exceeds twice the less by 18. Find the numbers.

2. If a certain number is increased by 16, the result is seven-thirds of the number. Find the given number.

3. A boy was asked how many marbles he had. He replied, "If you take away 8 from twice the number I have, and divide the remainder by 3, the result is just one-half the number." How many marbles had he?

4. The sum of the denominator and twice the numerator of a certain fraction is 26. If 3 is added to both numerator and denominator, the resulting fraction is $\frac{2}{3}$. Find the fraction.

5. A courier sent away with a despatch travels uniformly at the rate of 12 miles per hour; 2 hours after his departure a second courier starts to overtake the first, traveling uniformly at the rate of $13\frac{1}{2}$ miles per hour. In how many hours will the second courier overtake the first?

6. Solve Example 5 when the respective rates of the first and second couriers are a and b miles per hour, and the interval between their departures is c hours.

7. A certain railroad train travels at a uniform rate. If the rate were 6 miles per hour faster, the distance traveled in 8 hours would exceed by 50 miles the distance traveled in 11 hours at a rate 7 miles per hour less than the actual rate. Find the actual rate of the train.

8. A can do a piece of work in 10 days; A and B together can do it in 7 days. In how many days can B do it alone?

9. A can do a piece of work in a days; A and B together can do it in b days. In how many days can B do it alone?

10. If A can do a piece of work in $2m$ days, B and A together in n days, and A and C in $m+\frac{n}{2}$ days, how long will it take them to do the work together?

11. A boatman moves 5 miles in $\frac{3}{4}$ of an hour, rowing with the tide; to return it takes him $1\frac{1}{2}$ hours, rowing against a tide one-half as strong. What is the velocity of the stronger tide?

12. A boatman, rowing with the tide, moves a miles in b hours. Returning, it takes him c hours to accomplish the same distance, rowing against a tide m times as strong as the first. What is the velocity of the stronger tide?

13. If A, who is traveling, makes $\frac{1}{2}$ of a mile more per hour, he will be on the road only $\frac{4}{5}$ of the time; but if he makes $\frac{1}{2}$ of a mile less per hour, he will be on the road $2\frac{1}{2}$ hours more. Find the distance and the rate.

14. The circumference of a fore wheel of a carriage is a feet; that of a hind wheel, b feet. What distance will the carriage have passed over when a fore wheel has made n more revolutions than a hind wheel?

15. A is 72 years old, and B is two-thirds as old as A. How many years ago was A five times as old as B?

16. If three pipes can fill a cistern in a, b, and c minutes respectively, in how many minutes will the cistern be filled by the three pipes together?

17. Find the time between 2 and 3 o'clock when the hands of a clock are together.

18. Find the time between 3 and 4 o'clock when the hands of a clock make a right angle.

19. A merchant maintained himself for three years at an expense of $1500 a year, and each year increased that part of his stock that was not so expended by one-third of it. At the end of the third year his original stock was doubled. What was his original stock?

20. When a body of troops was formed into a solid square there were 60 men over; but when formed in a column with 5 men more in front than before and 3 men less in depth, 1 man more was needed to complete the column. Find the number of troops.

21. A man engaged to work a days on these conditions: for each day he worked he was to receive b cents, and for each day he was idle he was to forfeit c cents. At the end of a days he received d cents. How many days was he idle?

22. A banker has two kinds of coins. It takes a pieces of the first kind to make a dollar, and b pieces of the second to make a dollar. A person wishes to obtain c pieces for a dollar. How many pieces of each kind must the banker give him?

23. A wine merchant has two kinds of wine which cost him, one a dollars, and the other b dollars, per gallon. He wishes to make a mixture of l gallons, which shall cost him on the average m dollars a gallon. How many gallons must he take of each?

Discuss the question (i) when $a = b$; (ii) when a or $b = m$; (iii) when $a = b = m$; (iv) when $a > b$ and $< m$; (v) when $a > b$ and $b > m$.

CHAPTER VII

SIMULTANEOUS SIMPLE EQUATIONS

131. Equations that express *different* relations between the unknown numbers are called **independent equations.**

Thus, $x + y = 10$ and $x - y = 2$ are independent equations; they express *different* relations between x and y. But $x + y = 10$ and $3x + 3y = 30$ are not independent equations; both express the *same* relation between the unknown numbers.

132. An equation is said to be **satisfied** by a number, if we can substitute that number for one of the unknowns in the equation without destroying the equality.

133. Simultaneous equations are solved by combining the equations so as to obtain a single equation with one unknown number; this process is called **elimination.**

There are three methods of elimination in general use:

I. By Addition or Subtraction.
II. By Substitution.
III. By Comparison.

We shall give one example of each method.

(1) Solve $\left.\begin{aligned} 2x - 3y &= 4 \\ 3x + 2y &= 32 \end{aligned}\right\}.$ [1] [2]

Multiply [1] by 2, $4x - 6y = 8$ [3]

Multiply [2] by 3, $9x + 6y = 96$ [4]

Add [3] and [4], $13x = 104$

$\therefore x = 8.$

Substitute the value of x in [2], $24 + 2y = 32.$

$\therefore y = 4.$

In this solution y is eliminated by *addition.*

(2) Solve $\left.\begin{array}{l}2x+3y=8\\3x+7y=7\end{array}\right\}.$ [1] [2]

Transpose $3y$ in [1], $2x = 8 - 3y.$

Divide by coefficient of x, $x = \dfrac{8-3y}{2}.$ [3]

Substitute the value of x in [2],

$$3\left(\frac{8-3y}{2}\right) + 7y = 7.$$

$$\frac{24-9y}{2} + 7y = 7.$$

$$24 - 9y + 14y = 14.$$

$$5y = -10.$$

$$\therefore y = -2.$$

Substitute the value of y in [3], $\therefore x = 7.$

In this solution y is eliminated by *substitution*.

(3) Solve $\left.\begin{array}{l}2x-9y=11\\3x-4y=\ \ 7\end{array}\right\}.$ [1] [2]

Transpose $9y$ in [1], $2x = 11 + 9y.$ [3]

Transpose $4y$ in [2], $3x = 7 + 4y.$ [4]

Divide [3] by 2, $x = \dfrac{11+9y}{2}.$ [5]

Divide [4] by 3, $x = \dfrac{7+4y}{3}.$ [6]

Equate the values of x, $\dfrac{11+9y}{2} = \dfrac{7+4y}{3}.$ [7]

Reduce [7], $33 + 27y = 14 + 8y.$

$$19y = -19.$$

$$\therefore y = -1.$$

Substitute the value of y in [5], $\therefore x = 1.$

In this solution x is eliminated by *comparison*.

Each equation must be simplified, if necessary, before the elimination is performed.

(4) Solve
$$\left.\begin{aligned}(x-1)(y+2)&=(x-3)(y-1)+8\\ \frac{2x-1}{5}-\frac{3(y-2)}{4}&=1\end{aligned}\right\}. \qquad \begin{aligned}&[1]\\&[2]\end{aligned}$$

Simplify [1], $xy+2x-y-2=xy-x-3y+3+8.$

Transpose and combine, $3x+2y=13.$ [3]

Simplify [2], $8x-4-15y+30=20.$

Transpose and combine, $8x-15y=-6.$ [4]

Multiply [3] by 8, $24x+16y=104.$ [5]

Multiply [4] by 3, $24x-45y=-18.$ [6]

Subtract [6] from [5], $61y=122.$

$\therefore y=2.$

Substitute the value of y in [3], $3x+4=13.$

$\therefore x=3.$

Fractional simultaneous equations, with denominators which are simple expressions containing the unknown numbers, may be solved as follows:

(5) Solve
$$\left.\begin{aligned}\frac{5}{3x}+\frac{2}{5y}&=7\\ \frac{7}{6x}-\frac{1}{10y}&=3\end{aligned}\right\}. \qquad \begin{aligned}&[1]\\&[2]\end{aligned}$$

Multiply [2] by 4, $\frac{14}{3x}-\frac{2}{5y}=12.$ [3]

Add [1] and [3], $\frac{19}{3x}=19.$

Divide by 19, $\frac{1}{3x}=1.$

$\therefore x=\frac{1}{3}.$

Substitute the value of x in [1],

$$5+\frac{2}{5y}=7.$$

Transpose, $\frac{2}{5y}=2.$

Divide by 2, $\frac{1}{5y}=1.$

$\therefore y=\frac{1}{5}.$

134. Literal Simultaneous Equations. The method of solving literal simultaneous equations is as follows:

Solve
$$\left.\begin{aligned} ax + by &= m \\ cx + dy &= n \end{aligned}\right\}. \qquad \begin{matrix}[1]\\ [2]\end{matrix}$$

Multiply [1] by c, $\quad acx + bcy = cm.$ [3]

Multiply [2] by a, $\quad acx + ady = an.$ [4]

Subtract [4] from [3], $\quad (bc - ad)y = cm - an.$

Divide by coefficient of y, $\quad y = \dfrac{cm - an}{bc - ad}.$

Multiply [1] by d, $\quad adx + bdy = dm.$ [5]

Multiply [2] by b, $\quad bcx + bdy = bn.$ [6]

Subtract [6] from [5], $\quad (ad - bc)x = dm - bn.$

Divide by coefficient of x, $\quad x = \dfrac{dm - bn}{ad - bc}.$

135. If three simultaneous equations are given, involving three unknown numbers, one of the unknown numbers must be eliminated between *two pairs* of the equations; then a second between the resulting equations.

Solve
$$\left.\begin{aligned} 2x - 3y + 4z &= 4 \\ 3x + 5y - 7z &= 12 \\ 5x - y - 8z &= 5 \end{aligned}\right\}. \qquad \begin{matrix}[1]\\ [2]\\ [3]\end{matrix}$$

Eliminate z between two pairs of these equations.

Multiply [1] by 2, $\quad 4x - 6y + 8z = 8.$

[3] is $\quad 5x - y - 8z = 5.$

Add, $\quad 9x - 7y = 13.$ [4]

Multiply [1] by 7, $\quad 14x - 21y + 28z = 28.$

Multiply [2] by 4, $\quad 12x + 20y - 28z = 48.$

Add, $\quad 26x - y = 76.$ [5]

Multiply [5] by 7, $\quad 182x - 7y = 532.$ [6]

[4] is $\quad 9x - 7y = 13.$

Subtract [4] from [6], $\quad 173x = 519.$

$$\therefore x = 3.$$

Substitute the value of x in [5], $\quad 78 - y = 76.$

$$\therefore y = 2.$$

Substitute the values of x and y in [1], $\quad 6 - 6 + 4z = 4.$

$$\therefore z = 1.$$

Likewise, if four or more equations are given, involving four or more unknown numbers, we must eliminate one of the unknown numbers from three or more pairs of the equations, using every equation at least once; then a second unknown number from pairs of the resulting equations; and so on.

Exercise 13

Solve the following sets of equations:

1. $\left.\begin{array}{r} 6x+5y=46 \\ 10x+3y=66 \end{array}\right\}.$

2. $\left.\begin{array}{r} 2x+7y=52 \\ 3x-5y=16 \end{array}\right\}.$

3. $\left.\begin{array}{r} 4x+9y=79 \\ 7x-17y=40 \end{array}\right\}.$

4. $\left.\begin{array}{r} 2x-7y=8 \\ 4y-9x=19 \end{array}\right\}.$

5. $\left.\begin{array}{l} x=16-4y \\ y=34-4x \end{array}\right\}.$

6. $\left.\begin{array}{l} 5x=2y+78 \\ 3y=x+104 \end{array}\right\}.$

7. $\left.\begin{array}{l} \dfrac{2x}{3}+\dfrac{y}{2}=10 \\ \dfrac{y}{4}=\dfrac{5x-7}{19} \end{array}\right\}.$

8. $\left.\begin{array}{l} 4+y=\dfrac{3x}{4} \\ x-8=\dfrac{4y}{5} \end{array}\right\}.$

9. $\left.\begin{array}{l} a^2+ax+y=0 \\ b^2+bx+y=0 \end{array}\right\}.$

10. $\left.\begin{array}{l} \dfrac{x+y}{3}+x=15 \\ \dfrac{x-y}{5}+y=6 \end{array}\right\}.$

11. $\left.\begin{array}{l} \dfrac{x-1}{8}+\dfrac{y-2}{5}=2 \\ \dfrac{2x}{7}+\dfrac{2y-5}{21}=3 \end{array}\right\}.$

12. $\left.\begin{array}{l} \dfrac{3}{x}+\dfrac{8}{y}=3 \\ \dfrac{15}{x}-\dfrac{4}{y}=4 \end{array}\right\}.$

13. $\left.\begin{array}{l} \dfrac{4}{5x}+\dfrac{5}{6y}=\dfrac{86}{15} \\ \dfrac{5}{4x}-\dfrac{4}{5y}=\dfrac{11}{20} \end{array}\right\}.$

14. $\left.\begin{array}{l} \dfrac{x-2}{5}-\dfrac{10-x}{3}=\dfrac{y-10}{4} \\ \dfrac{2y+4}{3}=\dfrac{4x+y+13}{8} \end{array}\right\}.$

15. $\left.\begin{array}{l} (a+c)x+(a-c)y=2ab \\ (a+b)y-(a-b)x=2ac \end{array}\right\}.$

16. $$\left.\begin{aligned} x - \frac{2y - x}{23 - x} &= 20 + \frac{2x - 59}{2} \\ y - \frac{y - 3}{x - 18} &= 30 - \frac{73 - 3y}{3} \end{aligned}\right\}.$$

17. $$\left.\begin{aligned} 2x - 3y &= 5a - b \\ 3x - 2y &= 5a + b \end{aligned}\right\}.$$

18. $$\left.\begin{aligned} \frac{x}{a} + \frac{y}{b} &= 1 - \frac{x}{c} \\ \frac{x}{b} + \frac{y}{a} &= 1 + \frac{y}{c} \end{aligned}\right\}.$$

19. $$\left.\begin{aligned} \frac{x + y}{x - y} &= \frac{a}{b - c} \\ \frac{x + c}{y + b} &= \frac{a + b}{a + c} \end{aligned}\right\}.$$

20. $$\left.\begin{aligned} \frac{x - a}{y - a} &= \frac{a - b}{a + b} \\ \frac{x}{y} &= \frac{a^3 - b^3}{a^3 + b^3} \end{aligned}\right\}.$$

21. $$\left.\begin{aligned} 8x + 4y - 3z &= 6 \\ x + 3y - z &= 7 \\ 4x - 5y + 4z &= 8 \end{aligned}\right\}.$$

22. $$\left.\begin{aligned} x + \frac{y}{b} - \frac{z}{c} &= a \\ y + \frac{z}{c} - \frac{x}{a} &= b \\ z + \frac{x}{a} - \frac{y}{b} &= c \end{aligned}\right\}.$$

23. $$\left.\begin{aligned} 3x - \frac{y}{4} + z &= 7\tfrac{1}{2} \\ 2x - \frac{y - 3z}{3} &= 5\tfrac{1}{8} \\ 2x - \frac{y}{2} + 4z &= 11 \end{aligned}\right\}.$$

24. $$\left.\begin{aligned} \frac{3y - 2x}{3z - 7} &= \frac{1}{2} \\ \frac{5z - x}{2y - 3z} &= 1 \\ \frac{y - 2z}{3y - 2x} &= 1 \end{aligned}\right\}.$$

25. $$\left.\begin{aligned} \frac{a}{x} + \frac{b}{y} + \frac{c}{z} &= 3 \\ \frac{a}{x} + \frac{b}{y} - \frac{c}{z} &= 1 \\ \frac{2a}{x} &= \frac{b}{y} + \frac{c}{z} \end{aligned}\right\}.$$

26. $$\left.\begin{aligned} \frac{xy}{x + y} &= a \\ \frac{xz}{x + z} &= b \\ \frac{yz}{y + z} &= c \end{aligned}\right\}.$$

27. $$\left.\begin{aligned} (a + b)x + (b + c)y + (c + a)z &= ab + bc + ca \\ (a + c)x + (a + b)y + (b + c)z &= ab + ac + bc \\ (b + c)x + (a + c)y + (a + b)z &= a^2 + b^2 + c^2 \end{aligned}\right\}.$$

136. Problems. It is often necessary in the solution of problems to employ two or more letters to represent the numbers to be found. In all cases the conditions must be sufficient to give just as many equations as there are unknown numbers.

If there are *more* equations than unknown numbers, some are superfluous or inconsistent; if there are *fewer* equations than unknown numbers, the problem is indeterminate.

If A gives B \$10, B will have three times as much money as A. If B gives A \$10, A will have twice as much money as B. How much has each?

Let $x =$ the number of dollars A has,
and $y =$ the number of dollars B has.
Then $y + 10 =$ the number of dollars B has after receiving \$10,
$x - 10 =$ the number of dollars A has after giving \$10,
$x + 10 =$ the number of dollars A has after receiving \$10,
and $y - 10 =$ the number of dollars B has after giving \$10.

$\therefore$ $y + 10 = 3(x - 10)$,
and $x + 10 = 2(y - 10)$.

From the solution of these equations, $x = 22$ and $y = 26$.

Therefore, A has \$22 and B has \$26.

Exercise 14

1. Three times the greater of two numbers exceeds twice the less by 27; and the sum of twice the greater and five times the less is 94. Find the numbers.

2. A fraction is such that if 3 is added to each of its terms, the resulting fraction is equal to $\frac{4}{5}$; and if 3 is subtracted from each of its terms, the result is equal to $\frac{1}{2}$. Find the fraction.

3. Two women buy velvet and silk. One buys $3\frac{1}{2}$ yards of velvet and $12\frac{3}{4}$ yards of silk; the other buys $4\frac{1}{2}$ yards of velvet and 5 yards of silk. Each woman pays \$63.80. Find the price per yard of the velvet and of the silk.

4. Each of two persons owes \$1200. The first said to the second, "If you give me $\frac{3}{4}$ of what you have, I shall have enough to pay my debt." The second replied, "If you give me $\frac{5}{8}$ of what your purse contains, I can pay my debt." How much does each have?

5. Two passengers have together 400 pounds of baggage. One pays \$1.20, the other \$1.80, for excess above the weight allowed. If all the baggage had belonged to one person he would have paid \$4.50. How much baggage is allowed free?

6. A number is formed by two digits. The sum of the digits is 6 times their difference. The number itself exceeds 6 times the sum of its digits by 3. Find the number.

7. A number is formed by two digits of which the sum is 8. If the digits are interchanged, 4 times the new number exceeds the original number by 2 more than 5 times the sum of the digits. Find the original number.

8. Three brothers, A, B, C, have together bought a house for \$32,000. A could pay the whole sum if B would give him $\frac{5}{8}$ of what he has; B could pay it if C would give him $\frac{8}{9}$ of what he has; and C could pay the whole sum if he had $\frac{1}{2}$ of what A has together with $\frac{3}{16}$ of what B has. How much does each have?

9. A and B entered into partnership with a joint capital of \$3400. A put in his money for 12 months; B put in his money for 16 months. In closing the business, B's share of the profits was greater than A's by $\frac{5}{59}$ of the total profit. Find the sum put in by each.

10. A capitalist makes two investments; the first at 3 per cent, the second at $3\frac{1}{2}$ per cent. His total income from the two investments is \$427. If \$1400 was taken from the second investment and added to the first, the incomes from the two investments would be equal. Find the amount of each investment.

11. A cask contains 12 gallons of wine and 18 gallons of water; a second cask contains 9 gallons of wine and 3 gallons of water. How many gallons must be taken from each cask, so that, when mixed, there may be 14 gallons consisting half of water and half of wine?

12. A and B ran a race to a post and back. A returning meets B 30 yards from the post and beats him by 1 minute. If on arriving at the starting place A had immediately returned to meet B, he would have run $\frac{1}{6}$ the distance to the post before meeting him. Find the distance run, and the time A and B each makes.

13. A and B together can do a piece of work in 15 days. After working together for 6 days, A leaves off and B finishes the work in 30 days more. In how many days can each do the work?

14. A and B together can do a piece of work in 12 days. After working together 9 days, however, they call in C to aid them, and the three finish the work in 2 days. C finds that he can do as much work in 5 days as A does in 6 days. In how many days can each do the work?

15. A pedestrian has a certain distance to walk. After having passed over 20 miles, he increases his speed by 1 mile per hour. If he had walked the entire journey with this speed, he would have accomplished his walk in 40 minutes less time; but, by keeping his first place, he would have arrived 20 minutes later than he did. What distance had he to walk?

16. A person invests \$10,000 in three per cent bonds, \$16,500 in three and one-half per cents, and has an income from both investments of \$1056.25. If his investments had been \$2750 more in the three per cents, and less in the three and one-half per cents, his income would have been $62\frac{1}{2}$ cents greater. Find the price of each kind of bonds.

CHAPTER VIII

INVOLUTION AND EVOLUTION

137. Involution is the operation of raising an expression to any required *power*. (See § 15.)

Every case of involution is merely an example of *multiplication,* in which the factors are *equal.*

138. Index Law. If m is a positive integer, by definition

$$a^m = a \times a \times a \cdots \text{to } m \text{ factors.} \qquad (\S\ 16)$$

Consequently, if m and n are both positive integers,

$$\begin{aligned}(a^n)^m &= a^n \times a^n \times a^n \cdots \text{to } m \text{ factors}\\ &= (a \times a \cdots \text{to } n \text{ factors})(a \times a \cdots \text{to } n \text{ factors})\\ &\qquad \cdots \text{taken } m \text{ times}\\ &= a \times a \times a \cdots \text{to } mn \text{ factors}\\ &= a^{mn}.\end{aligned}$$

The above is the **index law** for involution.

Similarly,

$$(a^m)^n = a^{mn} = (a^n)^m.$$

Also,
$$\begin{aligned}(ab)^n &= ab \times ab \cdots \text{to } n \text{ factors}\\ &= (a \times a \cdots \text{to } n \text{ factors})(b \times b \cdots \text{to } n \text{ factors})\\ &= a^n b^n.\end{aligned}$$

139. If the exponent of the required power is a composite number, the exponent may be resolved into prime factors, the power denoted by one of these factors found, and the result raised to a power denoted by another factor; and so on.

Thus, the fourth power may be obtained by taking the second power of the second power; the sixth by taking the second power of the third power; and so on.

140. From the *Law of Signs* in multiplication it is evident that all *even* powers of a scalar number are *positive*; all *odd* powers of a scalar number have the *same sign* as the number itself. (§ 55)

The even powers of two compound expressions which have the same terms with opposite signs are identical.

Thus, $(b-a)^2 = \{-(a-b)\}^2 = (a-b)^2.$

141. Binomials. By actual multiplication we obtain

$$(a+b)^2 = a^2 + 2ab + b^2;$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3;$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

In these results it will be observed that:

1. The number of terms is greater by one than the exponent of the power to which the binomial is raised.

2. In the first term the exponent of a is the same as the exponent of the power to which the binomial is raised, and it decreases by one in each succeeding term.

3. b appears in the second term with 1 for an exponent, and its exponent increases by 1 in each succeeding term.

4. The coefficient of the first term is 1.

5. The coefficient of the second term is the same as the exponent of the power to which the binomial is raised.

6. The coefficient of each succeeding term is found from the next preceding term by multiplying the coefficient of that term by the exponent of a, and dividing the product by a number greater by 1 than the exponent of b.

If b is negative, the terms in which the *odd* powers of b occur are negative.

Thus, $(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$

By the above rules any power of a binomial of the form $a+b$ or $a-b$ may be written at once.

142. The same method may be employed when the terms of a binomial have *coefficients* or *exponents*.

(1) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

(2) $$\begin{aligned}(5x^2-2y^3)^3 &= (5x^2)^3 - 3(5x^2)^2(2y^3) + 3(5x^2)(2y^3)^2 - (2y^3)^3\\ &= 125x^6 - 150x^4y^3 + 60x^2y^6 - 8y^9.\end{aligned}$$

In like manner, a *polynomial* of three or more terms may be raised to any power by enclosing its terms in parentheses, so as to give the expression the form of a binomial.

(3) $$\begin{aligned}&(x^3-2x^2+3x+4)^2\\ &= \{(x^3-2x^2)+(3x+4)\}^2\\ &= (x^3-2x^2)^2 + 2(x^3-2x^2)(3x+4) + (3x+4)^2\\ &= x^6 - 4x^5 + 4x^4 + 6x^4 - 4x^3 - 16x^2 + 9x^2 + 24x + 16\\ &= x^6 - 4x^5 + 10x^4 - 4x^3 - 7x^2 + 24x + 16.\end{aligned}$$

Exercise 15

Perform the indicated operations:

1. $(2a^3)^4$.
2. $(3a^2x^3)^3$.
3. $\left(\dfrac{2a^4b^2}{3c^3x^4}\right)^5$.
4. $(-4b^2c)^3$.
5. $(-a^2b^3c)^4$.
6. $\dfrac{(3a^4b^2)^4}{(9a^5b^3)^3}$.
7. $(-5a^3b^2x^4)^3$.
8. $(6a^2b^3c^4)^4$.
9. $\dfrac{(-3a^3x^2)^5}{(6a^5bx^2)^3}$.
10. $\dfrac{(3a^2x^3)^2(4b^2x)^4}{(6b^3x^5)^2(a^2b)^2}$.
11. $\dfrac{(4x^4y)^3}{(9x^2y^3)^4} \div \dfrac{(x^2y^5)^2}{(2y)^5}$.
12. $(x+3)^3$.
13. $(1-2x)^4$.
14. $(3-x^2)^5$.
15. $(1-4x)^6$.
16. $\left(1-\dfrac{2x}{3}\right)^5$.
17. $\left(1+\dfrac{3x^2}{4}\right)^4$.
18. $\left(\dfrac{3}{2x}-\dfrac{2x}{3}\right)^5$.
19. $(1+3x)^7$.
20. $\left(\dfrac{2}{x}-\dfrac{3x}{4}\right)^6$.

21. $\left(2a^2b^m - \dfrac{a^mb^2}{2}\right)^5.$

22. $\left(x^{n-2} - \dfrac{y^{n-4}}{4x^n}\right)^4.$

23. $(1+3x-x^2)^4.$

24. $\left(1+\dfrac{x}{2}-\dfrac{x^2}{4}\right)^3.$

143. **Evolution** is the operation of resolving a number into factors all equal to one another. If a number is resolved into *two* equal factors, either factor is called a **square root** of the number; if into *three* equal factors, each factor is called a **cube root**; if into *four* equal factors, each factor is called a **fourth root**; and generally if a number is resolved into n equal factors, n being any positive integer, each of these factors is called an **nth root** of the number.

Under the term number *is included any algebraic expression* (§ 7), *whether monomial or polynomial, integral or fractional.*

The symbol which denotes that a square root is to be extracted is $\sqrt{\ }$; and for other roots the same symbol is used, but with a figure written above to indicate the root; thus, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, etc., signify the *third* root, *fourth* root, etc.

144. If k, m, and n are positive integers, we have

$$(a^m)^n = a^{mn}. \qquad (\S\ 138)$$

Therefore, a^m is an nth root of a^{mn}.

That is, $\qquad a^m = \text{one value of } \sqrt[n]{a^{mn}}.$

Also, § 138, $\qquad (a^kb^m)^n = a^{kn}b^{mn}.$

Consequently, $\qquad a^kb^m = \text{one value of } \sqrt[n]{a^{kn}b^{mn}}.$

Hence, a root of a monomial is found by *dividing the exponent of each factor by the index of the root and taking the product of the resulting factors,* first expressing the numerical coefficient, if other than 1, as a product of its prime factors. The root thus obtained is called the **principal root** of the monomial for the given index.

Thus, $3a^3$ is the principal fourth root of $81a^{12}$ $(= 3^4a^{12})$.

By the Law of Signs for Multiplication, § 55,

$$(+a) \times (+a) = +a^2,$$

and $$(-a) \times (-a) = +a^2.$$

Therefore, $\sqrt{+a^2}$ may be either $+a$ or $-a$,

but $\sqrt{-a^2}$ can be neither $+a$ nor $-a$.

Hence,

I. Every positive number has *two* square roots, equal in absolute value but opposite in sign, one being positive, the other negative. This is indicated by writing the double sign $\pm$ before the root, which sign is read *plus or minus*.

Hence, also, any even-indexed root of a positive number will have the double sign $\pm$.

II. No scalar number can be the square root of a negative number.

III. An odd-indexed root of a scalar number has the same sign as the number itself.

145. The indicated even root of a negative number is called an **imaginary** or **orthotomic** number.

146. **Square Roots of Compound Expressions.** Since the square of $a + b$ is $a^2 + 2ab + b^2$, the square root of $a^2 + 2ab + b^2$ is $a + b$.

It is required to devise a method for extracting the square root $a + b$ when $a^2 + 2ab + b^2$ is given.

The first term a of the root is obviously the square root of the first term a^2 of the expression.

$$\begin{array}{r|l} & a^2 + 2ab + b^2 \,(a + b \\ & a^2 \\ \hline 2a + b & 2ab + b^2 \\ & 2ab + b^2 \\ \hline \end{array}$$

If a^2 is subtracted from the given expression, the remainder is $2ab + b^2$. Therefore, the second term b of the root is obtained by dividing the first term of this remainder by $2a$, that is, by *double the part of the root already found.* Also, since $2ab + b^2 = (2a + b)b$, the divisor *is completed by adding to the trial-divisor the new term of the root.*

The same method applies to longer expressions, if care is taken to obtain the *trial-divisor* at each stage of the process

by ***doubling the part of the root already found***, and to obtain the ***complete divisor by annexing the new term of the root to the trial-divisor.***

Find the square root of

$$1 + 10x^2 + 25x^4 + 16x^6 - 24x^5 - 20x^3 - 4x.$$

Arrange according to ascending or descending powers of x. Thus,

$$\begin{array}{r|l}
 & 16x^6 - 24x^5 + 25x^4 - 20x^3 + 10x^2 - 4x + 1\,(4x^3 - 3x^2 + 2x - 1 \\
 & 16x^6 \\
\hline
8x^3 - 3x^2 & -24x^5 + 25x^4 \\
 & -24x^5 + 9x^4 \\
\hline
8x^3 - 6x^2 + 2x & 16x^4 - 20x^3 + 10x^2 \\
 & 16x^4 - 12x^3 + 4x^2 \\
\hline
8x^3 - 6x^2 + 4x - 1 & -8x^3 + 6x^2 - 4x + 1 \\
 & -8x^3 + 6x^2 - 4x + 1 \\
\hline
\end{array}$$

It will be noticed that each successive trial-divisor may be obtained by taking the preceding complete divisor with its ***last term doubled.***

147. Square Roots of Arithmetical Numbers. In extracting the square root of a number expressed by figures, the first step is to separate the figures into groups.

Since $1 = 1^2$, $100 = 10^2$, $10{,}000 = 100^2$, and so on, it is evident that the square root of any integral square number between 1 and 100 lies between 1 and 10; the square root of any integral square number between 100 and 10,000 lies between 10 and 100; and so on. In other words, the square root of any integral square number expressed by *one* or *two* figures is a number of *one* figure; the square root of any integral square number expressed by *three* or *four* figures is a number of *two* figures; and so on.

If, therefore, an integral square number is divided into groups of two figures each, from the right to the left, the number of figures in the root is equal to the number of groups of figures. The last group to the left may consist of only one figure.

Find the square root of 3249.

$$\begin{array}{r|r}
 & 32\ 49\,(57 \\
 & 25 \\
\hline
107 & 7\ 49 \\
 & 7\ 49 \\
\hline
\end{array}$$

In this case, a in the typical form $a^2 + 2ab + b^2$ represents 5 *tens*, that is, 50, and b represents 7. The 25 subtracted is really 2500, that is, a^2, and the complete divisor $2a + b$ is $2 \times 50 + 7 = 107$.

The same method applies to numbers of more than two groups by considering that a in the typical form represents at each step *the part of the root already found*, and that a represents *tens* with reference to the next figure of the root.

148. If the square root of a number has decimal places, the number itself has *twice* as many.

Thus, if 0.21 is the square root of some number, this number is

$$(0.21)^2 = 0.21 \times 0.21 = 0.0441.$$

Hence, if the given square number contains a decimal, we divide it into groups of two figures each, by beginning at the decimal point and proceeding toward the left for the integral number and toward the right for the decimal. We must be careful to have the last group on the right of the decimal point contain *two* figures, annexing a cipher when necessary.

Find the square root of 41.2164, and of 965.9664.

```
      41.21 64(6.42              9 65.96 64(31.08
      36                         9
  124)521                     61)65
       496                       61
 1282)25 64                 6208)496 64
      25 64                      496 64
```

149. If a number contains an *odd* number of decimal places, or gives a *remainder* when as many figures in the root have been obtained as the given number has groups, then its exact square root cannot be found. We may, however, approximate to the root as near as we please by annexing ciphers and continuing the operation.

Find the square root of 3, and of 357.357.

```
       3(1.732···               3 57.35 70(18.903···
       1                        1
   27)2 00                   28)2 57
      1 89                      2 24
  343)11 00                 369)33 35
      10 29                     33 21
 3462)71 00               37803)14 70 00
      69 24                     11 34 09
```

Exercise 16

Simplify:

1. $\sqrt{16\,a^8b^4}$.
3. $\sqrt[4]{81\,a^8b^{12}}$.
5. $\sqrt[5]{1024\,a^{10}b^5}$.
2. $\dfrac{\sqrt[3]{27\,a^9b^6}}{\sqrt{81\,a^4}}$.
4. $\dfrac{\sqrt[4]{625\,x^8}}{\sqrt[3]{64\,x^6}}$.
6. $\dfrac{\sqrt[3]{216\,a^6x^3}}{\sqrt[5]{32\,a^{15}x^{10}}}$.

Extract the square root of:

7. $1 + 4\,x + 10\,x^2 + 12\,x^3 + 9\,x^4$.
8. $9 - 24\,x - 68\,x^2 + 112\,x^3 + 196\,x^4$.
9. $4 - 12\,x + 5\,x^2 + 26\,x^3 - 29\,x^4 - 10\,x^5 + 25\,x^6$.
10. $36\,x^2 - 120\,a^2x - 12\,a^4x + 100\,a^4 + 20\,a^6 + a^8$.
11. $4 + 9\,y^2 - 20\,x + 25\,x^2 + 30\,xy - 12\,y$.
12. $4\,x^4 + 9\,y^6 - 12\,x^2y^3 + 16\,x^2 + 16 - 24\,y^3$.
13. $\dfrac{x^4}{4} + \dfrac{y^6}{9} + \dfrac{x^2}{4} - \dfrac{x^2y^3}{3} + \dfrac{1}{16} - \dfrac{y^3}{6}$.

Extract to four places of decimals the square root of:

14. 326.
15. 1020.
16. 3.666.
17. 1.31213.

150. Cube Roots of Compound Expressions. Since the cube of $a + b$ is $a^3 + 3\,a^2b + 3\,ab^2 + b^3$, the cube root of

$$a^3 + 3\,a^2b + 3\,ab^2 + b^3 \text{ is } a + b.$$

It is required to devise a method for extracting the cube root $a + b$ when $a^3 + 3\,a^2b + 3\,ab^2 + b^3$ is given.

Find the cube root of $a^3 + 3\,a^2b + 3\,ab^2 + b^3$.

$$\begin{array}{r|l}
 & a^3 + 3\,a^2b + 3\,ab^2 + b^3\,(a + b \\
3\,a^2 & a^3 \\
\cline{2-2}
+ 3\,ab + b^2 & 3\,a^2b + 3\,ab^2 + b^3 \\
\cline{1-1}
3\,a^2 + 3\,ab + b^2 & 3\,a^2b + 3\,ab^2 + b^3 \\
\cline{2-2}
\end{array}$$

The first term of the root is a the cube root of a^3.

If a^3 is subtracted, the remainder is $3a^2b + 3ab^2 + b^3$; therefore, the second term b of the root is obtained by dividing the first term of this remainder by *three times the square of* a.

Also, since $3a^2b + 3ab^2 + b^3 = (3a^2 + 3ab + b^2)b$, the *complete divisor* is obtained by adding $3ab + b^2$ to the *trial-divisor* $3a^2$.

The same method may be applied to longer expressions by considering a in the typical form $3a^2 + 3ab + b^2$ to represent at each stage of the process *the part of the root already found.*

Find the cube root of $x^6 - 3x^5 + 5x^3 - 3x - 1$.

$$
\begin{array}{rr|l}
 & & \underline{|\,x^2 - x - 1} \\
 & & x^6 - 3x^5 + 5x^3 - 3x - 1 \\
 & 3x^4 & x^6 \\
(3x^2 - x)(-x) = & -3x^3 + x^2 & -3x^5 + 5x^3 \\
 & 3x^4 - 3x^3 + x^2 & -3x^5 + 3x^4 - x^3 \\
 & & -3x^4 + 6x^3 - 3x - 1 \\
3(x^2 - x)^2 = & 3x^4 - 6x^3 + 3x^2 & \\
(3x^2 - 3x - 1)(-1) = & -3x^2 + 3x + 1 & \\
 & 3x^4 - 6x^3 + 3x + 1 & -3x^4 + 6x^3 - 3x - 1
\end{array}
$$

The first trial-divisor is $3x^4$, and the first complete divisor is $3x^4 - 3x^3 + x^2$. The second trial-divisor is $3(x^2 - x)^2$, or $3x^4 - 6x^3 + 3x^2$. The second term of the root is found by dividing $-3x^5$, the first term of the remainder, by $3x^4$, the first term of the first trial-divisor. The second complete divisor is $3x^4 - 6x^3 + 3x + 1$.

151. Cube Roots of Arithmetical Numbers. In extracting the cube root of a number expressed by figures, the first step is to separate the figures into groups.

Since $1 = 1^3$, $1000 = 10^3$, $1,000,000 = 100^3$, and so on, it follows that the cube root of any integral cube number between 1 and 1000, that is, of any integral cube number that has *one*, *two*, or *three* figures, is a number of *one* figure; and that the cube root of any integral cube number between 1000 and 1,000,000, that is, of any integral cube number that has *four*, *five*, or *six* figures, is a number of *two* figures; and so on.

If, therefore, an integral cube number is divided into groups of three figures each, from right to left, the number of figures in the root is equal to the number of groups. The last group to the left may consist of one, two, or three figures.

If the cube root of a number has decimal places, the number itself contains *three times* as many.

Hence, if a given number contains a decimal, we divide the figures into groups of three figures each, beginning at the decimal point and proceeding toward the left for the integral number, and toward the right for the decimal. We must annex ciphers if necessary, so that the last group on the right shall contain *three* figures.

If the given number is not a perfect cube, zeros may be annexed and an approximate value of the root found.

152. In the typical form, the *first complete divisor* is,

$$3a^2 + 3ab + b^2,$$

and the *second trial-divisor* is $3(a+b)^2$, that is,

$$3a^2 + 6ab + 3b^2,$$

which may be obtained from the preceding complete divisor by adding to it *its second term and twice its third term.*

Extract the cube root of 5 to five places of decimals.

	5.000(1.70997
	1
$3 \times 10^2 = 300$	4000
$3(10 \times 7) = 210$	
$7^2 = 49$	
559	3913
259	87000000
$3 \times 1700^2 = 8670000$	
$3(1700 \times 9) = 45900$	
$9^2 = 81$	
8715981	78443829
45981	85561710
$3 \times 1709^2 = 8762043$	78858387
	67033230
	61334301

After the first two figures of the root are found, the next trial-divisor is obtained by bringing down the sum of the 210 and 49 obtained in completing the preceding divisor; then adding the three lines connected by the brace, and annexing two ciphers to the result.

The last two figures of the root are found by division. The rule in such cases is, that two less than the number of figures already obtained may be found without error by division, the divisor to be employed being three times the square of the part of the root already found.

153. Since the fourth power is the square of the square, and the sixth power the square of the cube, the *fourth root* is the *square root* of the *square root*, and the *sixth root* is the *cube root* of the *square root*. In similar manner, the eighth, ninth, twelfth, $\cdots$ roots may be found.

Exercise 17

Extract the cube root of:

1. $27 - 108x + 144x^2 - 64x^3$.
2. $x^6 - 3x^5 + 5x^3 - 3x - 1$.
3. $a^3 - a^2b + \dfrac{ab^2}{3} - \dfrac{b^3}{27}$.
4. $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$.
5. $27 + 296x^3 - 125x^6 - 108x + 9x^2 - 15x^4 - 300x^5$.
6. $64x^6 + 192x^5 + 144x^4 - 32x^3 - 36x^2 + 12x - 1$.
7. $1 - 3x + 6x^2 - 10x^3 + 12x^4 - 12x^5 + 10x^6 - 6x^7 + 3x^8 - x^9$.
8. $a^6 - 12a^5b + 60a^4b^2 - 160a^3b^3 + 240a^2b^4 - 192ab^5 + 64b^6$.
9. $8a^6 + 48a^5b + 60a^4b^2 - 80a^3b^3 - 90a^2b^4 + 108ab^5 - 27b^6$.
10. $12x^2 - \dfrac{125}{x^3} - 54x - 59 + \dfrac{135}{x} + 8x^3 + \dfrac{75}{x^2}$.
11. $8x^3 - 36ax^2 + \dfrac{a^6}{x^3} + \dfrac{33a^4}{x} + 66a^2x - \dfrac{9a^5}{x^2} - 63a^3$.

Extract to three places of decimals the cube root of:

12. 517. 13. 1637. 14. 3.25. 15. 20.911.

CHAPTER IX

EXPONENTS

154. Positive Integral Exponents. If a is any definite number or any algebraic expression having one and only one value, and m and n are positive integers, we have, by the definitions of involution and evolution, §§ 15 and 19,

$$a^n = a \times a \times a \cdots \text{to } n \text{ factors},$$

and
$$(\sqrt[n]{a})^n = a.$$

We also know that
$$a^n = a^{n-1} \times a, \qquad (\S\ 56)$$

and
$$a^0 = 1. \qquad (\S\ 15)$$

We now easily deduce the following **Laws of Calculation:**

If a is any definite number or a single-valued algebraic expression and m and n are positive integers,

$$\text{I.}\quad \mathbf{a^m \div a^n = a^{m-n}},\ \text{if } \mathbf{n < m}, \text{ or if } \mathbf{n = m};$$

$$\text{II.}\quad \mathbf{a^m \div a^n = \frac{1}{a^{n-m}}},\ \text{if } \mathbf{n > m};$$

$$\text{III.}\quad \mathbf{(a^m)^n = a^{mn}};$$

$$\text{IV.}\quad \mathbf{(\sqrt[n]{a^m})^n = a^m}.$$

155. To obtain an interpretation of negative exponents we extend law I to include the case $n > m$; that is, we assume that law I holds true for all integral values of $m - n$, negative as well as positive, and interpret the result so that it shall be consistent with law II.

To obtain an interpretation of fractional exponents we extend law III to include all cases in which mn is integral; that is, we assume that law III holds true for all integral

values of n and mn, negative as well as positive, and so interpret the results that they shall be consistent with laws II and IV.

156. Negative Integral Exponents. If we divide a^3 successively by a in the ordinary manner, we have the series

$$a^3,\quad a^2,\quad a,\quad 1,\quad \frac{1}{a},\quad \frac{1}{a^2},\quad \frac{1}{a^3}. \qquad [1]$$

If we divide again by a by subtracting 1 from the exponent of the dividend, we have, since law II holds true, the series

$$a^3,\quad a^2,\quad a^1,\quad a^0,\quad a^{-1},\quad a^{-2},\quad a^{-3}. \qquad [2]$$

If we compare [1] and [2], we see that

$$a^0 = 1,\quad a^{-1} = \frac{1}{a},\quad a^{-2} = \frac{1}{a^2},\quad a^{-3} = \frac{1}{a^3}.$$

From the preceding we see at once that we may interpret a^{-n} as equivalent to $\frac{1}{a^n}$ consistently with law II.

Hence, $\qquad a^n = a \times a \times a \cdots$ to n factors;

and $\qquad a^{-n} = \frac{1}{a} \times \frac{1}{a} \times \frac{1}{a} \cdots$ to n factors.

157. Positive Fractional Exponents. If n is a positive integer, $\frac{1}{n}$ is a positive fraction.

We have, by the extended interpretation of law III,

$$\left(a^{\frac{1}{n}}\right)^n = a^{\frac{1}{n}\times n} = a^1 = a.$$

Taking the nth root of each side, we obtain

$$a^{\frac{1}{n}} = \sqrt[n]{a}; \qquad (\S\ 144)$$

that is, $a^{\frac{1}{n}}$ may be taken as denoting *any* number which when raised to the nth power produces a, and this is exactly what $\sqrt[n]{a}$ denotes. For example, $4^{\frac{1}{2}} = \sqrt{4} = \pm 2$.

Again, if m and n are both positive integers, by the extended interpretation of law III,

$$(a^{\frac{m}{n}})^n = a^{\frac{m}{n}\times n} = a^m;$$

but
$$(\sqrt[n]{a^m})^n = a^m. \quad \therefore a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

Hence, in a fractional exponent, the numerator indicates a power, and the denominator a root.

158. Negative Fractional Exponents. If n is a positive integer, $-\frac{1}{n}$ is a negative fraction, and we have, by the extended interpretations of laws I and III,

$$\left(a^{-\frac{1}{n}}\right)^n = a^{-\frac{1}{n}\times n} = a^{-1} = \frac{1}{a}.$$

Taking the nth root of each side, we obtain

$$a^{-\frac{1}{n}} = \frac{1}{\sqrt[n]{a}} = \frac{1}{a^{\frac{1}{n}}}. \qquad (\S\ 144)$$

Again, if m and n are both positive integers, by the extended interpretations of laws I and III,

$$\left(a^{-\frac{m}{n}}\right)^n = a^{-\frac{m}{n}\times n} = a^{-m} = \frac{1}{a^m}.$$

Taking the nth root of each side, we obtain

$$a^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{a^m}} = \frac{1}{a^{\frac{m}{n}}}.$$

Hence, whether the exponent is integral or fractional, we have always $a^{-m} = \frac{1}{a^m}.$

It is worthy of notice that while we have by definition

$$(a^{\frac{1}{n}})^n = a,$$

it does not necessarily follow that

$$(a^n)^{\frac{1}{n}} = a.$$

An example will make this plain.

$$(4^{\frac{1}{2}})^2 = (\pm 2)^2 = 4;$$

but

$$(4^2)^{\frac{1}{2}} = 16^{\frac{1}{2}} = \pm 4.$$

Hence, if $a^n = b^n$, it does not necessarily follow that $a = b$; all we are entitled to say is that if b takes in succession all possible values, one of these values must be a.

159. Index Laws. We shall reserve further discussion of this subject and the full and complete statement of the Index Laws to Chapter XXXIII. Meanwhile, **if we take into consideration only the principal values of all roots indicated,** we may enunciate the Index Laws as follows:

If a and b are single-valued expressions or numbers and m, n, r, s are any scalar integers, excluding zero values of m and n,

$$\text{I.}\quad \mathbf{a}^{\frac{r}{m}} \times \mathbf{a}^{\frac{s}{n}} = \mathbf{a}^{\left(\frac{r}{m}+\frac{s}{n}\right)}.$$

$$\text{II.}\quad (\mathbf{a}^{\frac{r}{m}})^{\frac{s}{n}} = \mathbf{a}^{\frac{rs}{mn}}.$$

$$\text{III.}\quad (\mathbf{ab})^{\frac{r}{m}} = \mathbf{a}^{\frac{r}{m}}\mathbf{b}^{\frac{r}{m}}.$$

160. Compound expressions are multiplied and divided as follows:

(1) Multiply $x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$ by $x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$.

$$\begin{array}{llll}
x^{\frac{1}{2}} & + x^{\frac{1}{4}}y^{\frac{1}{4}} & + y^{\frac{1}{2}} & \\
x^{\frac{1}{2}} & - x^{\frac{1}{4}}y^{\frac{1}{4}} & + y^{\frac{1}{2}} & \\
\hline
x & + x^{\frac{3}{4}}y^{\frac{1}{4}} & + x^{\frac{1}{2}}y^{\frac{1}{2}} & \\
 & - x^{\frac{3}{4}}y^{\frac{1}{4}} & - x^{\frac{1}{2}}y^{\frac{1}{2}} & - x^{\frac{1}{4}}y^{\frac{3}{4}} \\
 & & + x^{\frac{1}{2}}y^{\frac{1}{2}} & + x^{\frac{1}{4}}y^{\frac{3}{4}} + y \\
\hline
x & & + x^{\frac{1}{2}}y^{\frac{1}{2}} & + y
\end{array}$$

(2) Divide $\sqrt[3]{x^2} + \sqrt[3]{x} - 12$ by $\sqrt[3]{x} - 3$.

$$\begin{array}{l|l}
x^{\frac{2}{3}} + x^{\frac{1}{3}} - 12 & x^{\frac{1}{3}} - 3 \\
\cline{2-2}
x^{\frac{2}{3}} - 3x^{\frac{1}{3}} & x^{\frac{1}{3}} + 4 \\
\cline{1-1}
\quad + 4x^{\frac{1}{3}} - 12 & \\
\quad + 4x^{\frac{1}{3}} - 12 & \\
\cline{1-1}
\end{array}$$

Exercise 18

1. **Express with radical signs and positive exponents:**

$$a^{\frac{3}{4}};\quad b^{\frac{2}{5}};\quad c^{-\frac{1}{4}};\quad x^{-\frac{2}{3}};\quad (y^{\frac{1}{5}})^{-3}.$$

2. **Express with fractional exponents:**

$$\sqrt[3]{a^4};\quad \sqrt[5]{b^{12}};\quad \frac{1}{c\sqrt[5]{c^4}};\quad \sqrt{\frac{1}{x^{\frac{1}{3}}}};\quad \frac{1}{\sqrt[5]{y^3}}.$$

3. **Express with positive exponents:**

$$(a^{-3})^5;\quad \sqrt[4]{b^{-3}};\quad (\sqrt{c})^{-\frac{2}{3}};\quad \left(\frac{1}{\sqrt[4]{x^{-5}}}\right)^{-2}.$$

4. **Express without denominators:**

$$\frac{a^2}{(4x)^3};\quad \frac{a^{\frac{1}{2}}}{\sqrt{5x^3}};\quad \frac{4x^{-3}}{3y^{-2}};\quad \frac{2\sqrt{a^{-3}}}{3\sqrt[3]{x^5}}.$$

Simplify:

5. $a^{\frac{1}{2}} \times a^{-\frac{2}{3}} \times a^{-\frac{1}{6}};\quad b^{\frac{2}{3}} \times b^{\frac{1}{3}}\sqrt[4]{b^{-3}};\quad (\sqrt{c})^3\sqrt[3]{c^{-4}}.$

6. $a^{\frac{3}{4}} \times a^{\frac{1}{2}} \times \sqrt[3]{a^4};\quad b\sqrt[3]{c} \div (cx)^{\frac{1}{2}};\quad (a^{\frac{1}{2}}\sqrt[3]{ax})^{\frac{1}{6}}.$

7. $(3a)^{\frac{1}{2}}\sqrt{(16x)^3};\ \left(\dfrac{16a^{-4}}{81x^8}\right)^{-\frac{3}{4}};\ \left(\dfrac{9a^4}{16x^{-2}}\right)^{-\frac{3}{2}};\ \left(\dfrac{27a^3}{\sqrt{9x^4}}\right)^{-\frac{1}{3}}.$

Multiply:

8. $x^{\frac{4}{3}} - x^{\frac{2}{3}} + 1$ by $x^{\frac{2}{3}} + 1$.

9. $x^{2p} + x^py^p + y^{2p}$ by $x^{2p} - x^py^p + y^{2p}$.

10. $8a^{\frac{4}{3}} + 4a^{\frac{2}{3}}b^{-\frac{1}{3}} + 5a^{\frac{1}{3}}b^{-\frac{2}{3}} + 9b^{-\frac{4}{3}}$ by $2a^{\frac{1}{3}} - b^{-\frac{1}{3}}$.

Divide:

11. $x^{5n} + y^{5n}$ by $x^n + y^n$.

12. $x - y^{-4}$ by $x^{\frac{3}{4}} - x^{\frac{1}{2}}y^{-1} + x^{\frac{1}{4}}y^{-2} - y^{-3}$.

13. $a^{\frac{2}{3}} + b + c^{-\frac{1}{2}} - 3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{-\frac{1}{6}}$ by $a^{\frac{1}{3}} + b^{\frac{1}{2}} + c^{-\frac{1}{4}}$.

RADICAL EXPRESSIONS

161. An indicated root that can be obtained approximately but not exactly is called a **surd**.

The index of the required root shows the **order** of a surd; and surds are named quadratic, cubic, biquadratic, according as the second, third, or fourth roots are required.

The product of a rational factor and a surd factor is called a **mixed surd**; as, $3\sqrt{2}$, $b\sqrt{a}$.

When there is no rational factor outside of the radical sign, the surd is said to be **entire**; as, $\sqrt{2}$, $\sqrt{a}$.

162. Since $\sqrt[m]{a} \times \sqrt[m]{b} \times \sqrt[m]{c} = \sqrt[m]{abc}$, the product of two or more surds of the same order will be a radical expression of the same order, the number under the radical sign being the product of the numbers under the several radical signs.

In like manner, $\sqrt{a^2b} = \sqrt{a^2} \times \sqrt{b} = a\sqrt{b}$. That is,

A factor under the radical sign the root of which can be taken may, by having the root taken, be removed from under the radical sign.

Conversely, since $a\sqrt{b} = \sqrt{a^2b}$,

A factor outside the radical sign may be raised to the corresponding power and placed under the radical sign.

By $\sqrt[n]{a}$, where a is positive, is meant hereafter in this chapter the positive number which taken n times as a factor gives a for the product.

163. A surd is in its *simplest form* when the expression under the radical sign is integral and as small as possible.

Surds which, when reduced to the simplest form, have the same surd factor are said to be **similar**.

Simplify $\sqrt[3]{108}$; $\sqrt[5]{7x^2y^7}$.

$\sqrt[3]{108} = \sqrt[3]{27 \times 4} = 3\sqrt[3]{4}$. $\sqrt[5]{7x^2y^7} = \sqrt[5]{7x^2y^2 \times y^5} = y\sqrt[5]{7x^2y^2}$.

164. The product or quotient of two surds of *the same order* may be obtained by taking the product or quotient of the rational factors and of the surd factors separately.

Thus, $$2\sqrt{5} \times 5\sqrt{7} = 10\sqrt{35}.$$

Surds of *the same order* may be compared by expressing them as entire surds.

Compare $\frac{2}{3}\sqrt{7}$ and $\frac{3}{5}\sqrt{10}$.

$$\tfrac{2}{3}\sqrt{7} = \sqrt{\tfrac{28}{9}};$$
$$\tfrac{3}{5}\sqrt{10} = \sqrt{\tfrac{18}{5}}.$$
$$\sqrt{\tfrac{28}{9}} = \sqrt{\tfrac{140}{45}}, \text{ and } \sqrt{\tfrac{18}{5}} = \sqrt{\tfrac{162}{45}}.$$

As $\sqrt{\frac{162}{45}}$ is greater than $\sqrt{\frac{140}{45}}$, $\frac{3}{5}\sqrt{10}$ is greater than $\frac{2}{3}\sqrt{7}$.

165. The *order* of a surd may be changed by changing the *power* of the expression under the radical sign.

Thus, $$\sqrt{5} = \sqrt[4]{25}; \quad \sqrt[3]{c} = \sqrt[6]{c^2}.$$

Conversely, $$\sqrt[4]{25} = \sqrt{5}; \quad \sqrt[6]{c^2} = \sqrt[3]{c}.$$

In this way, surds of *different orders* may be reduced to the *same order* and may then be compared, multiplied, or divided.

(1) Compare $\sqrt{2}$ and $\sqrt[3]{3}$.

$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{3}{6}} = \sqrt[6]{2^3} = \sqrt[6]{8};$$
$$\sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{2}{6}} = \sqrt[6]{3^2} = \sqrt[6]{9}.$$

$\therefore$ $\sqrt[3]{3}$ is greater than $\sqrt{2}$.

(2) Multiply $\sqrt[3]{4a}$ by $\sqrt{6x}$.

$$\sqrt[3]{4a} = (4a)^{\frac{1}{3}} = (4a)^{\frac{2}{6}} = \sqrt[6]{(4a)^2} = \sqrt[6]{16a^2};$$
$$\sqrt{6x} = (6x)^{\frac{1}{2}} = (6x)^{\frac{3}{6}} = \sqrt[6]{(6x)^3} = \sqrt[6]{216x^3}.$$
$$\therefore \sqrt[3]{4a} \times \sqrt{6x} = \sqrt[6]{16a^2 \times 216x^3} = 2\sqrt[6]{54a^2x^3}.$$

(3) Divide $\sqrt[3]{3a}$ by $\sqrt{6b}$.

$$\sqrt[3]{3a} = (3a)^{\frac{1}{3}} = (3a)^{\frac{2}{6}} = \sqrt[6]{(3a)^2} = \sqrt[6]{9a^2};$$
$$\sqrt{6b} = (6b)^{\frac{1}{2}} = (6b)^{\frac{3}{6}} = \sqrt[6]{(6b)^3} = \sqrt[6]{216b^3}.$$
$$\therefore \sqrt[3]{3a} \div \sqrt{6b} = \sqrt[6]{\frac{9a^2}{216b^3}} = \frac{1}{6b}\sqrt[6]{1944a^2b^3}.$$

Exercise 19

Express as entire surds:

1. $3\sqrt{5}$; $5\sqrt{32}$; $a^2b\sqrt{bc}$; $3y^2\sqrt[4]{x^3y}$; $a^3\sqrt[4]{a^3b^2}$.

2. $5abc\sqrt{abc^{-1}}$; $\frac{2}{7}\sqrt{\frac{21}{8}}$; $(x+y)\sqrt{\frac{xy}{x^2+2xy+y^2}}$.

Express as mixed surds:

3. $\sqrt[3]{160x^4y^7}$; $\sqrt[3]{54x^2y^3}$; $\sqrt[4]{64x^5y^6}$; $\sqrt[3]{1372a^{15}b^{16}}$.

Simplify:

4. $2\sqrt[4]{80a^5b^2c^6}$; $7\sqrt{396x}$; $\sqrt{1\frac{1}{16}}$; $\sqrt{3\frac{1}{3}}$; $\sqrt{\frac{3a^2bx}{4cy^3}}$.

5. $\left(\frac{x^3y^2}{z^2}\right)\left(\frac{z^5}{x^5y^5}\right)^{\frac{1}{5}}$; $\left(\frac{a^3b^2}{c^4}\right)\left(\frac{c^9b^3}{a}\right)^{\frac{1}{3}}$; $(2a^2b^4)\times(b^2x^3)^{\frac{1}{2}}$.

6. Show that $\sqrt{20}$, $\sqrt{45}$, $\sqrt{\frac{4}{5}}$ are similar surds.

7. Show that $2\sqrt[3]{a^3b^2}$, $\sqrt[3]{8b^5}$, $\frac{1}{2}\sqrt[3]{\frac{a^6}{b}}$ are similar surds

8. Arrange in order of magnitude $9\sqrt{3}$, $6\sqrt{7}$, $5\sqrt{10}$.

9. Arrange in order of magnitude $4\sqrt[3]{4}$, $3\sqrt[3]{5}$, $5\sqrt[3]{3}$.

10. Multiply $3\sqrt{2}$ by $4\sqrt{6}$; $\frac{1}{2}\sqrt[3]{4}$ by $2\sqrt[3]{2}$.

11. Divide $2\sqrt{5}$ by $3\sqrt{15}$; $\frac{3}{8}\sqrt{21}$ by $\frac{9}{16}\sqrt{\frac{7}{20}}$.

12. Simplify $\frac{2\sqrt{10}}{3\sqrt{27}}\times\frac{7\sqrt{48}}{5\sqrt{14}}\div\frac{4\sqrt{15}}{15\sqrt{21}}$.

Arrange in order of magnitude:

13. $2\sqrt[3]{3}$, $3\sqrt{2}$, $\frac{4}{3}\sqrt[4]{4}$. 14. $3\sqrt{19}$, $5\sqrt[3]{2}$, $3\sqrt[3]{3}$.

Simplify:

15. $\sqrt[4]{a^2xy^3}\times\sqrt[5]{a^2xy}$; $3\sqrt[3]{4ab^2}+\sqrt{2a^3b}$.

16. $\sqrt{(\frac{16}{25})^7}\times\sqrt{(\frac{25}{64})^6}$; $(\sqrt[7]{a^3b})^3\times\sqrt[7]{(a^3b^{12})^4}$.

166. In the addition or the subtraction of surds each surd must be reduced to its simplest form; then, if the resulting surds are similar,

Add the rational factors, and to their sum annex the common surd factor.

If the resulting surds are not similar,

Connect them with their proper signs.

167. Operations with surds will be more easily performed if the arithmetical numbers contained in the surds are expressed in their prime factors, and if fractional exponents are used instead of radical signs.

(1) Simplify $\sqrt{27}+\sqrt{48}+\sqrt{147}$.

$$\sqrt{27}=(3^3)^{\frac{1}{2}}=3\times 3^{\frac{1}{2}}=3\sqrt{3};$$
$$\sqrt{48}=(2^4\times 3)^{\frac{1}{2}}=2^2\times 3^{\frac{1}{2}}=4\times 3^{\frac{1}{2}}=4\sqrt{3};$$
$$\sqrt{147}=(7^2\times 3)^{\frac{1}{2}}=7\times 3^{\frac{1}{2}}=7\sqrt{3}.$$
$$\therefore \sqrt{27}+\sqrt{48}+\sqrt{147}=(3+4+7)\sqrt{3}=14\sqrt{3}.$$

(2) Simplify $2\sqrt[3]{320}-3\sqrt[3]{40}$.

$$2\sqrt[3]{320}=2(2^6\times 5)^{\frac{1}{3}}=2\times 2^2\times 5^{\frac{1}{3}}=8\sqrt[3]{5};$$
$$3\sqrt[3]{40}=3(2^3\times 5)^{\frac{1}{3}}=3\times 2\times 5^{\frac{1}{3}}=6\sqrt[3]{5}.$$
$$\therefore 2\sqrt[3]{320}-3\sqrt[3]{40}=8\sqrt[3]{5}-6\sqrt[3]{5}=2\sqrt[3]{5}.$$

168. If we wish to find the approximate value of $\frac{3}{\sqrt{2}}$, it will save labor if we multiply both numerator and denominator by a factor that will render the denominator *rational*; in this case by $\sqrt{2}$.

Thus,
$$\frac{3}{\sqrt{2}}=\frac{3\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{3\sqrt{2}}{2}.$$

169. It is easy to rationalize the denominator of a fraction when that denominator is a *binomial* involving only quadratic surds. The factor required will consist of the terms of the given denominator, connected by a different sign.

Thus, $\dfrac{7-3\sqrt{5}}{6+2\sqrt{5}}$ will have its denominator rationalized if we multiply both terms of the fraction by $6-2\sqrt{5}$.

For, $\dfrac{7-3\sqrt{5}}{6+2\sqrt{5}}=\dfrac{(7-3\sqrt{5})(6-2\sqrt{5})}{(6+2\sqrt{5})(6-2\sqrt{5})}=\dfrac{72-32\sqrt{5}}{16}=\dfrac{9}{2}-2\sqrt{5}.$

170. By two operations the denominator of a fraction may be rationalized when that denominator consists of *three* quadratic surds.

Thus, if the denominator is $\sqrt{6}+\sqrt{3}-\sqrt{2}$, both terms of the fraction may be multiplied by $\sqrt{6}-\sqrt{3}+\sqrt{2}$. The resulting denominator will be $6-5+2\sqrt{6}=1+2\sqrt{6}$; and if both terms of the resulting fraction are multiplied by $1-2\sqrt{6}$, the denominator becomes $1-24$ or -23.

Exercise 20

Simplify:

1. $\sqrt{27}+2\sqrt{48}+3\sqrt{108}$; $7\sqrt[3]{54}+3\sqrt[3]{16}+\sqrt[3]{432}$.
2. $2\sqrt{3}+3\sqrt{1\frac{1}{3}}-\sqrt{5\frac{1}{3}}$; $2\sqrt{\frac{5}{3}}+\sqrt{60}-\sqrt{15}-\sqrt{\frac{3}{5}}$.
3. $\sqrt{\dfrac{a^4c}{b^3}}-\sqrt{\dfrac{a^2c^3}{bd^2}}-\sqrt{\dfrac{a^2cd^2}{bm^2}}$; $3\sqrt{\dfrac{2}{5}}+2\sqrt{\dfrac{1}{10}}-4\sqrt{\dfrac{1}{40}}$.
4. $2\sqrt[3]{40}+3\sqrt[3]{108}+\sqrt[3]{500}-\sqrt[3]{320}-2\sqrt[3]{1372}$.
5. $(\sqrt[9]{8})^4$; $(\sqrt[9]{27})^4$; $(\sqrt[3]{64})^3$; $(\sqrt[3]{4})^2$.
6. $(a\sqrt[3]{a})^{-3}$; $(x\sqrt[3]{x})^{-\frac{1}{2}}$; $(p^2\sqrt{p})^{\frac{1}{2}}$; $(a^{-3}\sqrt[4]{a^{-3}})^{-\frac{1}{3}}$.

Extract the square root of:

7. $x^{4m}+6x^{3m}y^n+11x^{2m}y^{2n}+6x^my^{3n}+y^{4n}$.
8. $1+4x^{-\frac{1}{3}}-2x^{-\frac{2}{3}}-4x^{-1}+25x^{-\frac{4}{3}}-24x^{-\frac{5}{3}}+16x^{-2}$.
9. $9x^{-4}-18x^{-3}y^{\frac{1}{2}}+15x^{-2}y-6x^{-1}y^{\frac{3}{2}}+y^2$.
10. Extract the cube root of

    $8x^3+12x^2-30x-35+45x^{-1}+27x^{-2}-27x^{-3}$.

Simplify:

11. $\left(x^{\frac{2}{3}}\sqrt{\left(\frac{x^{\frac{1}{2}}}{\sqrt[3]{x}}\right)^5}\right)^{\frac{12}{13}}.$

12. $\left(\frac{\sqrt[4]{a^3}\sqrt[3]{b^2}}{5\sqrt[4]{c^5}}\right)\left(\frac{2\,b^{\frac{1}{3}}}{5\,a\sqrt[12]{b^2c^9}}\right).$

13. $\left(\frac{10\sqrt[3]{a^2}}{\sqrt[4]{5\,b^{11}}}\right)\left(\frac{5\,a\sqrt[3]{a^2}}{4\,b\sqrt[5]{a^2}}\right).$

14. $\left(\frac{\sqrt[5]{a^3}\sqrt{b}}{c\sqrt[4]{b^3}}\right)\times\left(a\sqrt[5]{ab^5}\right).$

15. $\left(\frac{x^{p+q}}{x^q}\right)^p\left(\frac{x^{q-p}}{x^q}\right)^{p-q}.$

16. $\frac{x^{2p(q+1)}-y^{2q(p-1)}}{x^{p(q+1)}+y^{q(p-1)}}.$

17. $3(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2-4(a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}})+(a^{\frac{1}{2}}-2\,b^{\frac{1}{2}})^2.$

Find equivalent fractions with rational denominators for the following, and find their approximate values:

18. $\frac{3}{\sqrt{7}+\sqrt{5}}.$

19. $\frac{7}{2\sqrt{5}-\sqrt{6}}.$

20. $\frac{4-\sqrt{2}}{1+\sqrt{2}}.$

21. $\frac{6}{5-2\sqrt{6}}.$

22. $\frac{2}{\sqrt{3}}.$

23. $\frac{1}{\sqrt{5}-\sqrt{2}}.$

24. $\frac{7\sqrt{5}}{\sqrt{7}+\sqrt{3}}.$

25. $\frac{7-2\sqrt{3}+3\sqrt{2}}{3+3\sqrt{3}-2\sqrt{2}}.$

26. $\frac{3\sqrt{5}-4\sqrt{2}}{2\sqrt{5}+3\sqrt{2}}.$

27. $\frac{\sqrt{5}-\sqrt{6}}{2\sqrt{5}-\sqrt{6}}.$

28. $\frac{1}{\sqrt{5}+\sqrt{3}+\sqrt{7}}.$

29. $\frac{2}{\sqrt{5}-3\sqrt{2}+\sqrt{7}}.$

30. Extract the cube root of

$a^{-\frac{3}{2}}-6\,a^{-\frac{5}{4}}b^{-\frac{1}{4}}+15\,a^{-1}b^{-\frac{1}{2}}-20\,a^{-\frac{3}{4}}b^{-\frac{3}{4}}+15\,a^{-\frac{1}{2}}b^{-1}$
$-6\,a^{-\frac{1}{4}}b^{-\frac{5}{4}}+b^{-\frac{3}{2}}.$

CHAPTER X

QUADRATIC EQUATIONS

WE now resume the subject of equations where we left it at the end of Chapter VII. Having considered equations of the first degree with one or more unknowns, we come next to the consideration of quadratic equations.

171. A quadratic equation that involves but one unknown number can contain only:

1. Terms involving the square of the unknown number.
2. Terms involving the first power of the unknown number.
3. Terms which do not involve the unknown number.

If the similar terms are combined, every quadratic equation can be made to assume the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are known numbers, and x the unknown number.

If a, b, c are given numbers, the equation is a **numerical quadratic.** If a, b, c are numbers represented wholly or in part by letters, the equation is a **literal quadratic.**

Thus, $x^2 - 6x + 5 = 0$ is a numerical quadratic,
and $ax^2 + 2bx + 3c - ab = 0$ is a literal quadratic.

172. In the equation $ax^2 + bx + c = 0$, the numbers a, b, and c are called the **coefficients** of the equation. The third term c is called the **constant** term.

If the first power of x is wanting, the equation is a **pure quadratic**; in this case, $b = 0$.

If the first power of x is present, the equation is an **affected** or **complete quadratic.**

173. Solution of Pure Quadratic Equations.

(1) Solve the equation $5x^2 - 48 = 2x^2$.

We have	$5x^2 - 48 = 2x^2.$
Collect the terms,	$3x^2 = 48.$
Divide by 3,	$x^2 = 16.$
Extract the root,	$x = \pm 4.$

Observe that the roots are numerically equal, but one is positive and the other negative. There are but two roots, since there are but two square roots of any number.

It may seem as though we ought to write the sign $\pm$ before the x as well as before the 4. If we do this, we have

$$+x = +4, \quad -x = -4, \quad +x = -4, \quad -x = +4.$$

From the first and second, $x = 4$; from the third and fourth, $x = -4$; these values of x are both given by $x = \pm 4$. Hence, it is *unnecessary*, although *perfectly correct*, to write the $\pm$ sign on *both* sides of the reduced equation.

(2) Solve the equation $3x^2 - 15 = 0$.

We have	$3x^2 = 15,$
or	$x^2 = 5.$
Extract the root,	$x = \pm\sqrt{5}.$

The roots cannot be found exactly, since the square root of 5 cannot be found exactly; it can, however, be found as accurately as we please; for example, it lies between 2.23606 and 2.23607.

(3) Solve the equation $3x^2 + 15 = 0$.

We have	$3x^2 = -15,$
or	$x^2 = -5.$
Extract the root,	$x = \pm\sqrt{-5}.$

There is no scalar square root of a negative number, since any scalar number, positive or negative, multiplied by itself, gives a positive result.

174. A root that can be found exactly is called **an exact root** or **rational root**. Such roots are either whole numbers or fractions.

A root that is indicated but can be found only approximately is called a **surd root**. Such roots involve the roots of imperfect powers.

Exact and surd roots are together called **real roots**.

A root that is indicated but cannot be found as a number in the arithmetical scale, either positive or negative, is called an **imaginary root**. Such roots involve the even roots of negative numbers.

Exercise 21

Solve:

1. $\frac{x^2-5}{3}+\frac{2x^2+1}{6}=\frac{1}{2}.$

2. $\frac{3}{1+x}+\frac{3}{1-x}=8.$

3. $\frac{3}{4x^2}-\frac{1}{6x^2}=\frac{7}{3}.$

4. $5x^2-9=2x^2+24.$

5. $\frac{x^2}{5}-\frac{x^2-10}{15}=7-\frac{50+x^2}{25}.$

6. $\frac{3x^2-27}{x^2+3}+\frac{90+4x^2}{x^2+9}=7.$

7. $\frac{4x^2+5}{10}-\frac{2x^2-5}{15}=\frac{7x^2-25}{20}.$

8. $\frac{10x^2+17}{18}-\frac{12x^2+2}{11x^2-8}=\frac{5x^2-4}{9}.$

9. $x^2+bx+a=bx(1-bx).$

10. $ax^2+b=c.$

11. $x^2-ax+b=ax(x-1).$

12. $\frac{ab-x}{b-ax}=\frac{b-cx}{bc-x}.$

13. $\frac{3(x+a)}{4x-a}-\frac{2x+a}{2a+x}=1.$

14. $\dfrac{3a}{x-5a}+\dfrac{x+4a}{x+3a}=\dfrac{7a^2+2ax-x^2}{(x-5a)(x+3a)}$.

15. $\dfrac{2(a+2b)}{a+2x}+\dfrac{a-2x}{a+b}=\dfrac{b^2}{(a+b)(a+2x)}$.

175. Solution of Affected Quadratic Equations.

Since $(x \pm b)^2$ is identical with $x^2 \pm 2bx + b^2$, it is evident that the expression $x^2 \pm 2bx$ lacks only the third term b^2 of being a perfect square.

This third term is the square of half the coefficient of x.

Every affected quadratic may be made to assume the form $x^2 \pm 2bx = c$ by dividing the equation through by the coefficient of x^2 (§ 171).

To **solve** such an equation:

The first step is to add to both members *the square of half the coefficient of* x. This is called **completing the square.**

The second step is to *extract the square root* of each member of the resulting equation.

The third step is to *solve* the two resulting simple equations.

(1) Solve the equation $x^2 - 8x = 20$.

We have $x^2 - 8x = 20$.

Complete the square, $x^2 - 8x + 16 = 36$.

Extract the root, $x - 4 = \pm 6$.

Solve, $x = 4 + 6 = 10$,

or $x = 4 - 6 = -2$.

The roots are 10 and -2.

We write the $\pm$ sign on only one side of the equation, for the reason given after the first example of § 173.

Verify by putting these numbers for x in the given equation:

$x = 10$.	$x = -2$.
$10^2 - 8(10) = 20$,	$(-2)^2 - 8(-2) = 20$,
$100 - 80 = 20$.	$4 + 16 = 20$.

(2) Solve the equation $\dfrac{x+1}{x-1}=\dfrac{4x-3}{x+9}$.

Free from fractions, $(x+1)(x+9)=(x-1)(4x-3)$.

Simplify, $3x^2-17x=6$.

Divide by 3, $x^2-\frac{17}{3}x=2$.

Complete the square, $x^2-\frac{17}{3}x+\left(\frac{17}{6}\right)^2=\frac{361}{36}$.

Extract the root, $x-\frac{17}{6}=\pm\frac{19}{6}$.

Solve, $x=\frac{17}{6}+\frac{19}{6}=\frac{36}{6}=6$,

or $x=\frac{17}{6}-\frac{19}{6}=-\frac{2}{6}=-\frac{1}{3}$.

The roots are 6 and $-\frac{1}{3}$.

Verify by putting these numbers for x in the original equation:

$x=6.$	$x=-\frac{1}{3}.$
$\dfrac{6+1}{6-1}=\dfrac{24-3}{6+9},$	$\dfrac{-\frac{1}{3}+1}{-\frac{1}{3}-1}=\dfrac{-\frac{4}{3}-3}{-\frac{1}{3}+9},$
$\frac{7}{5}=\frac{21}{15}.$	$\dfrac{\frac{2}{3}}{-\frac{4}{3}}=\dfrac{-\frac{13}{3}}{\frac{26}{3}},$
	$-\frac{1}{2}=-\frac{13}{26}.$

176. When the coefficient of x^2 is not unity, we may proceed as in the preceding section, or we may complete the square by another method.

Since $(ax\pm b)^2$ is identical with $a^2x^2\pm 2abx+b^2$, it is evident that the expression $a^2x^2\pm 2abx$ lacks only the third term b^2 of being a perfect square.

This third term is the square of the quotient obtained by dividing the second term by twice the square root of the first term.

Every affected quadratic may be made to assume the form $a^2x^2\pm 2abx=c$ (§ 171).

To **solve** such an equation:

The first step is to *complete the square;* to do this, *we divide the second term by twice the square root of the first term, square the quotient, and add the result to each member of the equation.*

The second step is to *extract the square root* of each member of the resulting equation.

The third step is to *reduce* the two resulting simple equations.

177. Numerical quadratics are solved as follows:

(1) Solve the equation $16x^2 + 5x - 3 = 7x^2 - x + 45$.

$$16x^2 + 5x - 3 = 7x^2 - x + 45.$$

Simplify, $\quad 9x^2 + 6x = 48.$

Complete the square, $\quad 9x^2 + 6x + 1 = 49.$

Extract the root, $\quad 3x + 1 = \pm 7.$

Solve, $\quad 3x = -1 + 7 \text{ or } -1 - 7.$

$$\therefore 3x = 6 \text{ or } -8.$$

$$\therefore x = 2 \text{ or } -2\tfrac{2}{3}.$$

Verify by substituting 2 for x in the equation

$$16x^2 + 5x - 3 = 7x^2 - x + 45.$$

$$16(2)^2 + 5(2) - 3 = 7(2)^2 - (2) + 45,$$

$$64 + 10 - 3 = 28 - 2 + 45,$$

$$71 = 71.$$

Verify by substituting $-2\tfrac{2}{3}$ for x in the equation

$$16x^2 + 5x - 3 = 7x^2 - x + 45.$$

$$16(-\tfrac{8}{3})^2 + 5(-\tfrac{8}{3}) - 3 = 7(-\tfrac{8}{3})^2 - (-\tfrac{8}{3}) + 45,$$

$$\tfrac{1024}{9} - \tfrac{40}{3} - 3 = \tfrac{448}{9} + \tfrac{8}{3} + 45,$$

$$1024 - 120 - 27 = 448 + 24 + 405,$$

$$877 = 877.$$

(2) Solve the equation $3x^2 - 4x = 32$.

Since the exact root of 3, the coefficient of x^2, cannot be found, it is necessary to multiply or divide each term of the equation by 3 to make the coefficient of x^2 a *square number*.

Multiply by 3, $\quad 9x^2 - 12x = 96.$

Complete the square, $\quad 9x^2 - 12x + 4 = 100.$

Extract the root, $\quad 3x - 2 = \pm 10.$

Solve, $\quad 3x = 2 + 10 \text{ or } 2 - 10.$

$$\therefore 3x = 12 \text{ or } -8.$$

$$\therefore x = 4 \text{ or } -2\tfrac{2}{3}.$$

Or, divide by 3, $\quad x^2 - \dfrac{4x}{3} = \dfrac{32}{3}.$

Complete the square, $\quad x^2 - \dfrac{4x}{3} + \dfrac{4}{9} = \dfrac{32}{3} + \dfrac{4}{9} = \dfrac{100}{9}.$

Extract the root, $\quad x - \dfrac{2}{3} = \pm \dfrac{10}{3}.$

$$\therefore x = \frac{2 \pm 10}{3}$$
$$= 4 \text{ or } -2\tfrac{2}{3}.$$

Verify by substituting 4 for x in the original equation,

$$48 - 16 = 32,$$
$$32 = 32.$$

Verify by substituting $-2\frac{2}{3}$ for x in the original equation,

$$21\tfrac{1}{3} + 10\tfrac{2}{3} = 32,$$
$$32 = 32.$$

(3) Solve the equation $-3x^2 + 5x = -2$.

Since the *even* root of a *negative* number is impossible, it is necessary to change the sign of each term. The resulting equation is

$$3x^2 - 5x = 2.$$

Multiply by 3, $\quad 9x^2 - 15x = 6.$

Complete the square, $\quad 9x^2 - 15x + \frac{25}{4} = \frac{49}{4}.$

Extract the root, $\quad 3x - \frac{5}{2} = \pm \frac{7}{2}.$

Solve, $\quad 3x = \dfrac{5 \pm 7}{2}.$

$$\therefore 3x = 6 \text{ or } -1.$$
$$\therefore x = 2 \text{ or } -\tfrac{1}{3}.$$

Or, divide by 3, $\quad x^2 - \dfrac{5x}{3} = \dfrac{2}{3}.$

Complete the square, $\quad x^2 - \dfrac{5x}{3} + \dfrac{25}{36} = \dfrac{49}{36}.$

Extract the root, $\quad x - \frac{5}{6} = \pm \frac{7}{6}.$

$$\therefore x = \frac{5 \pm 7}{6} = 2 \text{ or } -\tfrac{1}{3}.$$

If the equation $3x^2 - 5x = 2$ is multiplied by *four times the coefficient of* x^2, fractions will be avoided.

$$36x^2 - 60x = 24.$$

Complete the square, $\quad 36x^2 - 60x + 25 = 49.$

Extract the root, $\quad 6x - 5 = \pm 7.$

Solve, $\quad 6x = 5 \pm 7.$

$$\therefore 6x = 12 \text{ or } -2.$$
$$\therefore x = 2 \text{ or } -\tfrac{1}{3}.$$

It will be observed that the number added to complete the square by this last method is *the square of the coefficient of* x in the original equation $3x^2 - 5x = 2$.

(4) Solve the equation $\dfrac{3}{5-x} - \dfrac{1}{2x-5} = 2.$

Simplify, $$4x^2 - 23x = -30.$$

Multiply by four times the coefficient of x^2, and add to each side the square of the coefficient of x,

$$64x^2 - (\ ) + (23)^2 = 529 - 480 = 49.$$

Extract the root, $$8x - 23 = \pm 7.$$

Solve, $$8x = 23 \pm 7.$$

$$\therefore 8x = 30 \text{ or } 16.$$

$$\therefore x = 3\tfrac{3}{4} \text{ or } 2.$$

If a trinomial is a perfect square, its root is found by taking the roots of the *first* and *third* terms and connecting them by the *sign* of the middle term. It is not necessary, therefore, in completing the square, to write the middle term, but its place may be indicated as in this example.

(5) Solve the equation $72x^2 - 30x = -7.$

Since $72 = 2^3 \times 3^2$, if the equation is multiplied by 2, the coefficient of x^2 in the resulting equation, $144x^2 - 60x = -14$, is a square number, and the term required to complete the square is $(\frac{60}{24})^2 = (\frac{5}{2})^2 = \frac{25}{4}$.

Hence, if the original equation is multiplied by 4×2, the coefficient of x^2 in the result is a square number, and fractions are avoided in the work.

Multiply the given equation by 8,

$$576x^2 - 240x = -56.$$

Complete the square, $$576x^2 - (\ ) + 25 = -31.$$

Extract the root, $$24x - 5 = \pm\sqrt{-31}.$$

Solve, $$24x = 5 \pm \sqrt{-31}.$$

$$\therefore x = \tfrac{1}{24}(5 \pm \sqrt{-31}).$$

NOTE. In solving the following equations care must be taken to select the method best adapted to the example under consideration.

Exercise 22

Solve:

1. $x^2 - 2x = 15.$
2. $x^2 - 14x = -48.$
3. $x^2 - x = 12.$
4. $x^2 - 3x = 28.$
5. $x^2 - 13x + 42 = 0.$
6. $x^2 - 21x + 108 = 0.$
7. $2x^2 + x = 6.$
8. $4x^2 + 7x = 15.$

9. $3x^2 - 19x + 28 = 0.$

10. $4x^2 + 17x - 15 = 0.$

11. $6x^2 - x = 12.$

12. $5x^2 - \frac{28}{3}x + 4 = 0.$

13. $6x^2 - 7x + \frac{5}{3} = 0.$

14. $\dfrac{x^2+1}{17} + (x+1)(x+2) = 0.$

15. $(x-5)^2 + x^2 - 5 = 16(x+3).$

16. $\dfrac{x^2}{6} + \dfrac{3x-19}{3} = \dfrac{11+x}{3}.$

17. $\dfrac{2x^2-11}{2x+3} = \dfrac{x+1}{2}.$

18. $\dfrac{x+1}{x} + \dfrac{x}{6} = \dfrac{11}{2x}.$

19. $\dfrac{x^2-4}{3x} + \dfrac{2x}{5} = x + \dfrac{1-2x}{5}.$

20. $x + \dfrac{x+6}{x-6} = 2(x-2).$

21. $\dfrac{6}{2x-6} + \dfrac{x}{3-x} = \dfrac{8}{x}.$

22. $\dfrac{x+2}{x-1} - \dfrac{4-x}{2x} = \dfrac{7}{3}.$

23. $\dfrac{x-6}{x-2} + \dfrac{x+5}{2x+1} = 1.$

24. $\dfrac{x-3}{x+4} + \dfrac{x-4}{2(x-1)} = \dfrac{1}{2}.$

25. $\dfrac{x+1}{x^2-4} + \dfrac{1-x}{x+2} = \dfrac{2}{5(x-2)}.$

26. $\dfrac{x-5}{x+3} + \dfrac{x-8}{x-3} = \dfrac{80}{x^2-9} + \dfrac{1}{2}.$

27. $\dfrac{1}{x-3} + \dfrac{7}{x+3} = \dfrac{14}{x^2-9} - \dfrac{x-4}{x+3}.$

28. $\dfrac{3x+5}{x+3} + \dfrac{x+3}{x-3} = \dfrac{x-1}{x^2-9}.$

29. $\dfrac{x+1}{x-1} + \dfrac{x+2}{x-2} = \dfrac{2x+13}{x+1}.$

30. $\dfrac{2x-1}{x+1} + \dfrac{3x-1}{x+2} + \dfrac{7-x}{x-1} = 4.$

31. $\dfrac{3x+2}{1-5x} + \dfrac{x-7}{1+5x} + \dfrac{6(x^2-x+1)}{25x^2-1} + 5 = 0.$

32. $\dfrac{x+7}{9-4x^2} - \dfrac{1-x}{2x+3} = \dfrac{4}{2x-3}.$

33. $\dfrac{2x+1}{x+3} + \dfrac{2(x+1)}{x+2} = 2\tfrac{1}{15}.$

178. **Literal quadratics are solved as follows:**

(1) Solve the equation $ax^2 + bx + c = 0$.

Transpose, $$ax^2 + bx = -c.$$

Multiply the equation by $4a$ and add the square of b,

$$4a^2x^2 + (\ ) + b^2 = b^2 - 4ac.$$

Extract the root, $$2ax + b = \pm\sqrt{b^2 - 4ac}.$$

Solve, $$2ax = -b \pm \sqrt{b^2 - 4ac}.$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

(2) Solve the equation $adx - acx^2 = bcx - bd$.

Transpose bcx and change the signs,

$$acx^2 + bcx - adx = bd.$$

Express the left member in *two terms*,

$$acx^2 + (bc - ad)x = bd.$$

Multiply by $4ac$,

$$4a^2c^2x^2 + 4ac(bc - ad)x = 4abcd.$$

Complete the square,

$$4a^2c^2x^2 + (\ ) + (bc - ad)^2 = b^2c^2 + 2abcd + a^2d^2.$$

Extract the root, $$2acx + (bc - ad) = \pm(bc + ad).$$

Solve, $$2acx = -(bc - ad) \pm (bc + ad).$$

$$\therefore 2acx = 2ad \text{ or } -2bc.$$

$$\therefore x = \frac{d}{c} \text{ or } -\frac{b}{a}.$$

(3) Solve the equation $px^2 - px + qx^2 + qx = \dfrac{pq}{p+q}$.

Express the left member in *two terms*,

$$(p+q)x^2 - (p-q)x = \frac{pq}{p+q}.$$

Multiply by 4 times the coefficient of x^2,

$$4(p+q)^2x^2 - 4(p^2 - q^2)x = 4pq.$$

Complete the square,

$$4(p+q)^2x^2 - (\ ) + (p-q)^2 = p^2 + 2pq + q^2.$$

Extract the root, $2(p+q)x - (p-q) = \pm(p+q)$.

Solve, $$2(p+q)x = (p-q) \pm (p+q).$$

$$\therefore 2(p+q)x = 2p \text{ or } -2q.$$

$$\therefore x = \frac{p}{p+q} \text{ or } -\frac{q}{p+q}.$$

Observe that the left member of the simplified equation must be expressed in two terms, simple or compound, the first term involving x^2, *the second involving* x.

Exercise 23

Solve:

1. $x^2 - 2ax = 3a^2$.
2. $x^2 + 7a^2 = 8ax$.
3. $4x(x-a) + a^2 = b^2$.
4. $\frac{x^2}{2} - \frac{ax}{3} = 2a(x+2a)$.
5. $x^2 = ax + b$.
6. $\frac{(x+a)^2}{a^2} = \frac{(x-a)^2}{b^2}$.
7. $x^2 - \frac{x}{a} = \frac{3}{4a^2}$.
8. $x^2 - (a+b)x = -ab$.
9. $x^2 - \frac{m^2+n^2}{mn}x + 1 = 0$.
10. $\frac{2x(a-x)}{3a-2x} = \frac{a}{4}$.
11. $2x^2 + \frac{ab}{2} = (a+b)x$.
12. $(x+m)^2 + (x-m)^2 = 5mx$.
13. $ax^2 + 5a^2x + \frac{9a^3}{4} = 0$.
14. $b(a-x)^2 = (b-1)x^2$.
15. $\frac{x}{a-x} + \frac{a+b}{x} = \frac{a}{a-x}$.
16. $\frac{x^2-ab}{x-b} = \frac{x+a}{2}$.
17. $\frac{a+b}{x-2a} + \frac{2a+b}{a} = \frac{x}{a}$.
18. $\frac{ax}{b^2} + \frac{a+x}{x} = \frac{5a+x}{2b}$.
19. $\frac{ab}{ax-bx} = a+b-(a-b)x$.
20. $\frac{5ab-3b^2-ax}{2a-x} = \frac{2a+x}{3}$.

21. $x^2 - ax = \dfrac{(3a+2x)b}{2} + \dfrac{3(a^2+b^2)}{4}$.

22. $\dfrac{3a}{x+a} + \dfrac{2a}{x+2a} = \dfrac{4a}{x} + \dfrac{a}{x+3a}$.

23. $\dfrac{a-b+x}{a+b+x} + \dfrac{a+b}{x+b} = 2$.

24. $\dfrac{a+4b}{x+2b} - \dfrac{a-4b}{x-2b} = \dfrac{4b}{a}$.

25. $\dfrac{4(x+a)}{a+b} - \dfrac{3(a+b)}{x+a} = 4$.

26. $\dfrac{(4a^2-9b^2)(x^2+1)}{4a^2+9b^2} = 2x$.

27. $(3a^2+b^2)(x^2-x+1) = (a^2+3b^2)(x^2+x+1)$.

28. $\dfrac{4a^2}{x+2} - \dfrac{b^2}{x-2} = \dfrac{4a^2-b^2}{x(4-x^2)}$.

29. $\dfrac{a+2b}{a-2b} = \dfrac{a^2}{(a-2b)x} - \dfrac{4b^2}{x^2}$.

30. $\dfrac{x+1}{c} - \dfrac{2}{cx} = \dfrac{x+2}{ax-bx}$.

31. $\dfrac{a-c}{x-a} - \dfrac{x-a}{a-c} = \dfrac{3b(x-c)}{(a-c)(x-a)}$.

32. $x(x+b^2-b) = ax(a+1) - (a+b)^2(a-b)$.

33. $\dfrac{x}{2} + \dfrac{(4m^2-n^2)mn}{x} = \dfrac{4m^2+n^2}{2}$.

34. $\dfrac{x^2}{m+n} - \left(1+\dfrac{1}{mn}\right)x + \dfrac{1}{m} + \dfrac{1}{n} = 0$.

35. $\dfrac{2ab}{3x+1} + \dfrac{(3x-1)b^2}{2x+1} = \dfrac{(2x+1)a^2}{3x+1}$.

36. $\dfrac{x+2a-4b}{2bx} - \dfrac{8b-7a}{ax-2bx} + \dfrac{x-4a}{2(ab-2b^2)} = 0$.

37. $\dfrac{1}{a+2b} - \dfrac{x}{a^2-4b^2} + \dfrac{x-5b}{(a+2b)x} = \dfrac{x+19b-2a}{2bx-ax}$.

38. $\dfrac{a-2b}{x+2b}+\dfrac{2(x+4a+3b)}{x-5a+3b}=0.$

39. $\dfrac{x+3b}{8a^2-12ab}+\dfrac{3b}{4a^2-9b^2}=\dfrac{a+3b}{(2a+3b)(x-3b)}.$

40. $\dfrac{1}{2x^2+x-1}+\dfrac{1}{2x^2-3x+1}=\dfrac{a}{2bx-b}+\dfrac{2bx+b}{a-ax^2}.$

41. $\dfrac{1}{x}+\dfrac{4ax^2+3b(2-x)}{2ax^2+2a+3b}=2.$

42. $\dfrac{x-a}{2b(x+a)}+\dfrac{2(ab-ax+2b^2)}{a(x+a)^2}=\dfrac{1}{a}.$

43. $\dfrac{2ax+b}{ax+b}+\dfrac{2ax-b}{ax-b}=\dfrac{9b^2x^2+(4a^2-6b^2)x-(a^2+b^2)}{a^2x^2-b^2}.$

44. $\dfrac{x+a+b}{x-3a+b}+\dfrac{3(a+c)}{x+b+c}=2.$

179. Solutions by a Formula. Every affected quadratic may be reduced to the form $x^2+px+q=0$, in which p and q represent numbers, positive or negative, integral or fractional.

Solve $x^2+px+q=0$.

Complete the square, $4x^2+(\;)+p^2=p^2-4q.$

Extract the root, $2x+p=\pm\sqrt{p^2-4q}.$

$$\therefore x=-\frac{p}{2}\pm\frac{1}{2}\sqrt{p^2-4q}.$$

By this formula the values of x in an equation of the form $x^2+px+q=0$ may be written at once.

Thus, take the equation

$$3x^2-5x+2=0.$$

Divide by 3, $x^2-\frac{5}{3}x+\frac{2}{3}=0.$

Here, $p=-\frac{5}{3}$, and $q=\frac{2}{3}$.

$$\begin{aligned}\therefore x&=\tfrac{5}{6}\pm\tfrac{1}{2}\sqrt{\tfrac{25}{9}-\tfrac{8}{3}}\\&=\tfrac{5}{6}\pm\tfrac{1}{6}\\&=1\text{ or }\tfrac{2}{3}.\end{aligned}$$

180. Solutions by Factoring. A quadratic which has been reduced to its simplest form, and has all its terms written on one side, may often have that side resolved into factors *by inspection.*

In this case the roots are seen at once without completing the square.

(1) Solve $x^2 + 7x - 60 = 0$.

Since $\quad x^2 + 7x - 60 \equiv (x + 12)(x - 5)$,
the equation $\quad x^2 + 7x - 60 = 0$
may be written $\quad (x + 12)(x - 5) = 0$.

It will be observed that if *either* of the factors $x + 12$ or $x - 5$ is 0, the *product of the two factors* is 0, and the equation is satisfied.

Hence, $\quad x + 12 = 0$, or $x - 5 = 0$.

$$\therefore x = -12, \text{ or } x = 5.$$

(2) Solve $x^2 + 7x = 0$.

The equation $\quad x^2 + 7x = 0$
becomes $\quad x(x + 7) = 0$,
and is satisfied $\quad$ if $x = 0$, or if $x + 7 = 0$.

Therefore, the roots are 0 and -7.

It will be observed that this method is easily applied to an equation *all* the terms of which contain x.

(3) Solve $2x^3 - x^2 - 6x = 0$.

The equation $\quad 2x^3 - x^2 - 6x = 0$
becomes $\quad x(2x^2 - x - 6) = 0$,
and is satisfied if $\quad x = 0$, or if $2x^2 - x - 6 = 0$.

By solving $2x^2 - x - 6 = 0$ the two roots 2 and $-\frac{3}{2}$ are found.
Therefore, the equation has *three* roots, 0, 2, $-\frac{3}{2}$.

(4) Solve $x^3 + x^2 - 4x - 4 = 0$.

The equation $\quad x^3 + x^2 - 4x - 4 = 0$
becomes $\quad x^2(x + 1) - 4(x + 1) = 0$,
or $\quad (x^2 - 4)(x + 1) = 0$.

Therefore, the roots of the equation are $-1, 2, -2$.

(5) Solve $x^3 - 2x^2 - 11x + 12 = 0$.

By trial we find that $x - 1$ is a factor of the left member (§ 87).

The given equation may be written

$$(x-1)(x^2-x-12)=0,$$

or

$$(x-1)(x+3)(x-4)=0.$$

Therefore, the roots are 1, 4, -3.

(6) Solve the equation $x(x^2-9)=a(a^2-9)$.

If we put a for x, the equation is satisfied; therefore a is a root (§ 87).

Transpose all the terms to the left member and divide by $x-a$.

The given equation may be written

$$(x-a)(x^2+ax+a^2-9)=0,$$

and is satisfied if $x-a=0$, or if $x^2+ax+a^2-9=0$.

The roots are found to be

$$a,\quad \frac{-a+\sqrt{36-3a^2}}{2},\quad \frac{-a-\sqrt{36-3a^2}}{2}.$$

Exercise 24

Find all the roots of:

1. $(x-1)(x-2)(x^2-4x+8)=0$.
2. $(x^2-2x+2)(x^2-6x+7)=0$.
3. $x^3+27=0$.
4. $x^4-81=0$.
5. $x^3-27+4(x^2-9)=0$.
6. $x^4+9x^2-16(x^2+9)=0$.
7. $2x^3+3x^2-2x-3=0$.
8. $x^4-4x^3+8x^2-32x=0$.
9. $x^3-x-6=0$.
10. $x^3-6x^2+11x-6=0$.
11. $x^4-3x^3-8x^2+6x+4=0$.
12. $x^3+x^2-14x-24=0$.
13. $x^4-6x^3+9x^2+4x-12=0$.

14. $x(x-3)(x+1)=a(a-3)(a+1)$.

15. $x(x-3)(x+1)=20$.

16. $(x-1)(x-2)(x-3)=24$.

17. $(x+2)(x-3)(x+4)=240$.

18. $(x+1)(x+5)(x-6)=96$.

181. Character of the Roots. Every quadratic equation can be made to assume the form

$$ax^2+bx+c=0.$$

Solving this equation, § 178, Example (1), we obtain for its two roots

$$\frac{-b+\sqrt{b^2-4\,ac}}{2\,a},\quad \frac{-b-\sqrt{b^2-4\,ac}}{2\,a}.$$

There are two roots, and only two roots, since there are two, and only two, square roots of the expression $b^2-4\,ac$.

As regards the character of the two roots, there are three cases to be distinguished:

1. $b^2-4\,ac$ *positive.* In this case the roots are *real* and *different.* That the roots are different appears by writing them as follows:

$$-\frac{b}{2\,a}+\frac{\sqrt{b^2-4\,ac}}{2\,a},\quad -\frac{b}{2\,a}-\frac{\sqrt{b^2-4\,ac}}{2\,a};$$

these expressions cannot be equal since $b^2-4\,ac$ is not zero.

If $b^2-4\,ac$ is a perfect square, the roots are rational. If $b^2-4\,ac$ is not a perfect square, the roots are surds.

2. $b^2-4\,ac=0$. In this case the two roots are *real* and *equal,* since they both become $-\dfrac{b}{2\,a}$.

3. $b^2-4\,ac$ *negative.* In this case both roots have a real part and an imaginary part and are called *imaginary roots.*

If we write them in the form

$$-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}, \quad -\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a},$$

we see that two imaginary roots of a quadratic cannot be equal, since b^2-4ac is not zero. They have the same real part, $-\frac{b}{2a}$, and the same imaginary parts with opposite signs. Such expressions are called *conjugate expressions.*

The above cases may also be distinguished as follows:

1. **$b^2-4ac>0$, roots real and different.**
2. **$b^2-4ac=0$, roots real and equal.**
3. **$b^2-4ac<0$, roots imaginary.**

182. By calculating the value of b^2-4ac we can determine the character of the roots of a given equation without solving the equation.

(1) $x^2-5x+6=0$.

Here $\qquad a=1,\quad b=-5,\quad c=6.$
Therefore, $\qquad b^2-4ac=25-24=1.$

The roots are real and different, and rational.

(2) $3x^2+7x-1=0$.

Here $\qquad a=3,\quad b=7,\quad c=-1.$
Therefore, $\qquad b^2-4ac=49+12=61.$

The roots are real and different, and are both surds.

(3) $4x^2-12x+9=0$.

Here $\qquad a=4,\quad b=-12,\quad c=9.$
Therefore, $\qquad b^2-4ac=144-144=0.$

The roots are real and equal.

(4) $2x^2-3x+4=0$.

Here $\qquad a=2,\quad b=-3,\quad c=4.$
Therefore, $\qquad b^2-4ac=9-32=-23.$

The roots are both imaginary.

(5) Find the values of m for which the equation

$$2\,mx^2 + (5\,m + 2)\,x + (4\,m + 1) = 0$$

has its two roots equal.

Here $\qquad a = 2\,m; \quad b = 5\,m + 2; \quad c = 4\,m + 1.$

If the roots are to be equal, we must have $b^2 - 4\,ac = 0$, or

$$(5\,m + 2)^2 - 8\,m\,(4\,m + 1) = 0.$$

The solution of this equation gives $m = 2$ or $-\frac{2}{7}$.

For these values of m the equation becomes

$$4\,x^2 + 12\,x + 9 = 0, \text{ and } 4\,x^2 - 4\,x + 1 = 0,$$

each of which has its roots equal.

Exercise 25

Determine, without solving, the character of the roots of each of the following equations:

1. $x^2 - 6\,x + 8 = 0.$
2. $x^2 - 4\,x + 2 = 0.$
3. $x^2 + 6\,x + 13 = 0.$
4. $4\,x^2 - 12\,x + 7 = 0.$
5. $5\,x^2 - 9\,x + 6 = 0.$
6. $16\,x^2 - 56\,x + 49 = 0.$
7. $3\,x^2 - 2\,x + 12 = 0.$
8. $2\,x^2 - 19\,x + 17 = 0.$
9. $9\,x^2 + 30\,x + 25 = 0.$
10. $17\,x^2 - 12\,x + \frac{36}{17} = 0.$

Determine the values of m for which the two roots of each of the following equations are equal:

11. $(3\,m + 1)\,x^2 + (2\,m + 2)\,x + m = 0.$
12. $(m - 2)\,x^2 + (m - 5)\,x + 2\,m - 5 = 0.$
13. $2\,mx^2 + x^2 - 6\,mx - 6\,x + 6\,m + 1 = 0.$
14. $mx^2 + 2\,x^2 + 2\,m = 3\,mx - 9\,x + 10.$

183. Problems involving Quadratics. Problems that involve quadratic equations apparently have two solutions, as a quadratic equation has two roots. When both roots are positive integers they will give two solutions.

Fractional and negative roots will in some problems give solutions; in other problems they will not give solutions.

No difficulty will be found in selecting the result which belongs to the particular problem we are solving.

Sometimes, by a change in the statement of the problem, we may form a new problem which corresponds to the result that was inapplicable to the original problem.

Imaginary roots will in some problems give solutions. Their interpretation in such cases will be given in Chapter XXXIII.

(1) The sum of the squares of two consecutive numbers is 481. Find the numbers.

Let $x =$ one number,
and $x + 1 =$ the other.
Then $x^2 + (x + 1)^2 = 481$,
or $2x^2 + 2x + 1 = 481$.

The solution of which gives $x = 15$ or $= -16$.

The positive root 15 gives for the numbers, 15 and 16.

The negative root -16 is inapplicable to the problem, as *consecutive numbers* are understood to be integers which follow one another in the common scale, 1, 2, 3, 4, . . .

(2) What is the price of eggs per dozen when 2 more in a shilling's worth lowers the price 1 penny per dozen?

Let $x =$ the number of eggs for a shilling.
Then $\frac{1}{x} =$ the cost of 1 egg in shillings,
and $\frac{12}{x} =$ the cost of 1 dozen in shillings.
But if $x + 2 =$ the number of eggs for a shilling,
then $\frac{12}{x+2} =$ the cost of 1 dozen in shillings.

$$\therefore \frac{12}{x} - \frac{12}{x+2} = \frac{1}{12} \text{ (1 penny being } \tfrac{1}{12} \text{ of a shilling).}$$

The solution of which gives $x = 16$ or -18.

And, if 16 eggs cost a shilling, 1 dozen will cost $\frac{12}{16}$ of a shilling, or 9 pence.

Therefore, the price of the eggs is 9 pence per dozen.

If the problem is changed so as to read: What is the price of eggs per dozen when 2 *less* in a shilling's worth *raises* the price 1 penny per dozen? the algebraic statement is

$$\frac{12}{x-2}-\frac{12}{x}=\frac{1}{12}.$$

The solution of this equation gives $x = 18$ or -16.

Hence, the number 18, which had a negative sign and was inapplicable in the original problem, is here the true result, while the -16 is inapplicable in this problem.

Exercise 26

1. The product of two consecutive numbers exceeds their sum by 181. Find the numbers.

2. The square of the sum of two consecutive numbers exceeds the sum of their squares by 220. Find the numbers.

3. The difference of the cubes of two consecutive numbers is 817. Find the numbers.

4. The difference of two numbers is 5 times the less, and the square of the less is twice the greater. Find the numbers.

5. The numerator of a certain fraction exceeds the denominator by 1. If the numerator and denominator are interchanged, the sum of the resulting fraction and the original fraction is $2\frac{1}{30}$. Find the original fraction.

6. The denominator of a certain fraction exceeds twice the numerator by 3. If $3\frac{3}{14}$ is added to the fraction, the resulting fraction is the reciprocal of the original fraction. Find the original fraction.

7. A farmer bought a number of geese for \$24. Had he bought 2 more geese for the same money, he would have paid $\frac{2}{5}$ of a dollar less for each. How many geese did he buy, and what did he pay for each?

State the problem to which the negative solution applies.

8. A laborer worked a number of days and received for his labor \$36. Had his wages been 20 cents more per day, he would have received the same amount for 2 days' less labor. What were his daily wages, and how many days did he work?

State the problem to which the negative solution applies.

9. For a journey of 336 miles, 4 days less would have sufficed had I traveled 2 miles more per day. How many days did the journey take?

State the problem to which the negative solution applies.

10. A farmer hires a number of acres for \$420. He lets all but 4 acres for \$420, and receives for each acre \$2.50 more than he pays for it. How many acres does he hire?

11. A broker sells a number of railway shares for \$3240. A few days later, the price having fallen \$9 a share, he buys, for the same sum, 5 more shares than he had sold. Find the number of shares transferred on each day, and the price paid.

12. A man bought a number of sheep for \$300. He kept 15 and sold the remainder for \$270, gaining half a dollar on each sheep sold. How many sheep did he buy, and what did he pay for each?

13. The length of a rectangular lot exceeds its breadth by 20 yards. If each dimension is increased by 20 yards, the area of the lot will be doubled. Find the dimensions of the lot.

14. Twice the breadth of a rectangular lot exceeds the length by 2 yards; the area of the lot is 1200 square yards. Find the length and the breadth.

15. Three times the breadth of a rectangular field, the area of which is 2 acres, exceeds twice the length by 8 rods. At \$5 per rod, what will it cost to fence the field?

16. Two pipes running together fill a cistern in $10\frac{2}{7}$ hours; the larger pipe will fill the cistern in 6 hours less time than the smaller pipe. How long will it take each pipe, running alone, to fill the cistern?

17. Three workmen, A, B, and C, dig a ditch. A can dig it alone in 6 days more time, B in 30 days more time, than the time it takes the three to dig the ditch together; C can dig the ditch in 3 times the time the three dig it in. How many days does it take the three, working together, to dig the ditch?

18. A cistern with a capacity of 900 gallons can be filled by two pipes running together in as many hours as the larger pipe brings in gallons per minute; the smaller pipe brings in per minute 1 gallon less than the larger pipe. How long will it take each pipe by itself to fill the cistern?

19. A number is formed by two digits, the second being less by 3 than one-half the square of the first. If 9 is added to the number, the order of the digits is reversed. Find the number.

20. A number is formed by two digits; 5 times the second digit exceeds the square of the first digit by 4. If 3 times the first digit is added to the number, the order of the digits is reversed. Find the number.

21. A boat's crew row 8 miles down a river and back again in 1 hour and 15 minutes. Their rate in still water is 3 miles per hour faster than twice the rate of the current. Find the rate of the crew and the rate of the current.

22. A jeweller sold a watch for \$22.75 and lost on the cost of the watch as many per cent as the watch cost dollars. What was the cost of the watch?

23. A farmer sold a horse for \$138 and gained on the cost $\frac{1}{8}$ as many per cent as the horse cost dollars. Find the cost of the horse.

24. A broker bought a number of $100 shares, when they were a certain per cent below par, for $8500. He afterwards sold all but 20, when they were the same per cent above par, for $9200. How many shares did he buy, and what did he pay for each share?

25. A drover bought a number of sheep for $110; 4 having died, he sold the remainder for $7.33⅓ a head and made on his investment 4 times as many per cent as he paid dollars for each sheep bought. How many sheep did he buy, and how many dollars did he make?

26. A certain train leaves A for B, distant 216 miles; 3 hours later another train leaves A to travel over the same route; the second train travels 8 miles per hour faster than the first, and arrives at B 45 minutes behind the first. Find the time each train takes to travel over the route.

27. A coach, due at B 12 hours after it leaves A, after traveling from A as many hours as it travels miles per hour, breaks down; it then proceeds at a rate 1 mile per hour less than half its former rate and arrives at B 3 hours late. Find the distance from A to B.

28. Several boys spent each the same sum of money. If there had been 5 boys more and each boy had spent 25 cents less, the amount spent by the boys would have been $37.50. If there had been 5 boys less and each boy had spent 25 cents more, the amount spent would have been $30. Find the number of boys and the amount each boy spent.

29. A detachment from an army was marching in regular column with 5 men more in depth than in front. On approaching the enemy the front was increased by 845 men, and the whole detachment was thus drawn up in 5 lines. Find the number of men.

CHAPTER XI

SIMULTANEOUS QUADRATIC EQUATIONS

QUADRATIC equations that involve two unknown numbers require different methods for their solution according to the form of the equations.

184. CASE I. When from one of the equations the value of one of the unknown numbers can be found in terms of the other, and this value *substituted* in the other equation.

Solve $$\left.\begin{aligned}3x^2 - 2xy &= 5\\ x - y &= 2\end{aligned}\right\}. \qquad \begin{matrix}[1]\\ [2]\end{matrix}$$

Transpose x in [2], $y = x - 2$.
Substitute in [1], $3x^2 - 2x(x-2) = 5$.
The solution of which gives $x = 1$ or -5.
$\therefore y = -1$ or -7.

Special methods often give more elegant solutions than the general method by substitution.

1. *When equations have the form* $x \pm y = a$, *and* $xy = b$; $x^2 \pm y^2 = a$, *and* $xy = b$; *or*, $x \pm y = a$, *and* $x^2 + y^2 = b$.

(1) Solve $$\left.\begin{aligned}x + y &= 40\\ xy &= 300\end{aligned}\right\}. \qquad \begin{matrix}[1]\\ [2]\end{matrix}$$

Square [1], $x^2 + 2xy + y^2 = 1600$. [3]
Multiply [2] by 4, $4xy = 1200$. [4]
Subtract [4] from [3], $x^2 - 2xy + y^2 = 400$. [5]
Extract the root, $x - y = \pm 20$. [6]
Add [6] and [1], $2x = 60$ or 20.
Subtract [6] from [1], $2y = 20$ or 60.

$$\therefore \left.\begin{aligned}x &= 30\\ y &= 10\end{aligned}\right\} \text{ or } \left.\begin{aligned}x &= 10\\ y &= 30\end{aligned}\right\}.$$

(2) Solve
$$\left.\begin{aligned} x - y &= 4 \\ x^2 + y^2 &= 40 \end{aligned}\right\}. \qquad \begin{matrix}[1]\\ [2]\end{matrix}$$

Square [1], $x^2 - 2xy + y^2 = 16.$ [3]

Subtract [2] from [3], $-2xy = -24.$ [4]

Subtract [4] from [2], $x^2 + 2xy + y^2 = 64.$

Extract the root, $x + y = \pm 8.$ [5]

Combine [5] and [1],
$$\left.\begin{aligned} x &= 6 \\ y &= 2 \end{aligned}\right\} \text{ or } \left.\begin{aligned} x &= -2 \\ y &= -6 \end{aligned}\right\}.$$

(3) Solve
$$\left.\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{9}{20} \\ \frac{1}{x^2} + \frac{1}{y^2} &= \frac{41}{400} \end{aligned}\right\}. \qquad \begin{matrix}[1]\\ [2]\end{matrix}$$

Square [1], $\dfrac{1}{x^2} + \dfrac{2}{xy} + \dfrac{1}{y^2} = \dfrac{81}{400}.$ [3]

Subtract [2] from [3], $\dfrac{2}{xy} = \dfrac{40}{400}.$ [4]

Subtract [4] from [2], $\dfrac{1}{x^2} - \dfrac{2}{xy} + \dfrac{1}{y^2} = \dfrac{1}{400}.$

Extract the root, $\dfrac{1}{x} - \dfrac{1}{y} = \pm \dfrac{1}{20}.$ [5]

Combine [1] and [5],
$$\left.\begin{aligned} x &= 4 \\ y &= 5 \end{aligned}\right\} \text{ or } \left.\begin{aligned} x &= 5 \\ y &= 4 \end{aligned}\right\}.$$

2. *When one equation may be simplified by dividing it by the other.*

(4) Solve
$$\left.\begin{aligned} x^3 + y^3 &= 91 \\ x + y &= 7 \end{aligned}\right\}. \qquad \begin{matrix}[1]\\ [2]\end{matrix}$$

Divide [1] by [2], $x^2 - xy + y^2 = 13.$ [3]

Square [2], $x^2 + 2xy + y^2 = 49.$ [4]

Subtract [3] from [4], $3xy = 36.$

Divide by -3, $-xy = -12.$ [5]

Add [5] and [3], $x^2 - 2xy + y^2 = 1.$

Extract the root, $x - y = \pm 1.$ [6]

Combine [6] and [2],
$$\left.\begin{aligned} x &= 4 \\ y &= 3 \end{aligned}\right\} \text{ or } \left.\begin{aligned} x &= 3 \\ y &= 4 \end{aligned}\right\}.$$

185. CASE II. When each of the two equations is *homogeneous* and of the *second degree*.

Solve
$$\left.\begin{aligned} 2y^2 - 4xy + 3x^2 &= 17 \\ y^2 - x^2 &= 16 \end{aligned}\right\}. \qquad \begin{matrix}[1]\\ [2]\end{matrix}$$

Let $y = vx$, and substitute vx for y in each equation.

From [1], $\quad 2v^2x^2 - 4vx^2 + 3x^2 = 17.$

$$\therefore x^2 = \frac{17}{2v^2 - 4v + 3}.$$

From [2], $\quad v^2x^2 - x^2 = 16.$

$$\therefore x^2 = \frac{16}{v^2 - 1}.$$

Equate the values of x^2,

$$\frac{17}{2v^2 - 4v + 3} = \frac{16}{v^2 - 1},$$

$$32v^2 - 64v + 48 = 17v^2 - 17,$$

$$15v^2 - 64v = -65.$$

The solution gives $\quad v = \frac{13}{5}$ or $\frac{5}{3}$.

$v = \frac{13}{5}$,	$v = \frac{5}{3}$,
$y = vx = \frac{13x}{5}$.	$y = vx = \frac{5x}{3}$.
Substitute in [2],	Substitute in [2],
$\frac{169x^2}{25} - x^2 = 16$,	$\frac{25x^2}{9} - x^2 = 16$,
$x^2 = \frac{25}{9}$,	$x^2 = 9$,
$x = \pm\frac{5}{3}$,	$x = \pm 3$,
$y = \frac{13x}{5} = \pm\frac{13}{3}$.	$y = \frac{5x}{3} = \pm 5$.

186. CASE III. When the two equations are *symmetrical* with respect to x and y.

In this case the general rule is to combine the equations in such a manner as to remove the highest powers of x and y.

(1) Solve
$$\left.\begin{aligned} x^3 + y^3 &= 18xy \\ x + y &= 12 \end{aligned}\right\}. \qquad \begin{matrix}[1]\\ [2]\end{matrix}$$

Divide [1] by [2], $\quad x^2 - xy + y^2 = \frac{3xy}{2}.$ $\qquad$ [3]

To remove x^2 and y^2, square [2],

$$x^2 + 2xy + y^2 = 144. \qquad [4]$$

Subtract [4] from [3], $\quad -8xy = \frac{3xy}{2} - 144,$

which gives $\quad xy = 32.$

We now have $\left.\begin{aligned} x + y &= 12 \\ xy &= 32 \end{aligned}\right\}.$

Solving as in Case I, we find, $\left.\begin{aligned} x &= 8 \\ y &= 4 \end{aligned}\right\}$ or $\left.\begin{aligned} x &= 4 \\ y &= 8 \end{aligned}\right\}.$

(2) Solve

$$\left.\begin{aligned} x^4 + y^4 &= 337 \qquad [1] \\ x + y &= 7 \qquad [2] \end{aligned}\right\}.$$

To remove x^4 and y^4, raise [2] to the fourth power,

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 2401. \qquad [3]$$

Subtract [1] from [3], $\quad 4x^3y + 6x^2y^2 + 4xy^3 = 2064.$

Divide by 2, $\quad 2x^3y + 3x^2y^2 + 2xy^3 = 1032. \qquad [4]$

Square [2] and multiply the result by $2xy$,

$$2x^3y + 4x^2y^2 + 2xy^3 = 98xy. \qquad [5]$$

Subtract [5] from [4], $\quad -x^2y^2 = 1032 - 98xy,$

or $\quad x^2y^2 - 98xy = -1032.$

This is a quadratic equation, with xy for the unknown number.

Solving, we find $\quad xy = 12$ or 86.

We now have to solve the two pairs of equations,

$$\left.\begin{aligned} x + y &= 7 \\ xy &= 12 \end{aligned}\right\}, \qquad \left.\begin{aligned} x + y &= 7 \\ xy &= 86 \end{aligned}\right\}.$$

From the first, $\left.\begin{aligned} x &= 4 \\ y &= 3 \end{aligned}\right\}$ or $\left.\begin{aligned} x &= 3 \\ y &= 4 \end{aligned}\right\}.$

From the second, $\left.\begin{aligned} x &= \frac{7 \pm \sqrt{-295}}{2} \\ y &= \frac{7 \mp \sqrt{-295}}{2} \end{aligned}\right\}.$

The preceding cases are *general methods* for the solution of equations that belong to the kinds referred to; often, however, in the solution of these and other kinds of simultaneous equations involving quadratics, a little ingenuity will suggest some step by which the roots may be found more easily than by the general method.

Exercise 27

1. $\left.\begin{aligned} x + y &= 8 \\ xy &= 15 \end{aligned}\right\}.$

2. $\left.\begin{aligned} x + y &= 6 \\ xy + 27 &= 0 \end{aligned}\right\}.$

3. $\left.\begin{aligned} x - y &= 5 \\ xy &= 24 \end{aligned}\right\}.$

4. $\left.\begin{aligned} x - y &= 16 \\ xy + 60 &= 0 \end{aligned}\right\}.$

5. $\left.\begin{aligned} x + 2y &= 12 \\ xy &= 18 \end{aligned}\right\}.$

6. $\left.\begin{aligned} 2x + 3y &= 1 \\ xy + 15 &= 0 \end{aligned}\right\}.$

7. $\left.\begin{aligned} y &= 9 - 3x \\ x^2 &= 10 - xy \end{aligned}\right\}.$

8. $\left.\begin{aligned} x + 2y &= 12 \\ xy + y^2 &= 35 \end{aligned}\right\}.$

9. $\left.\begin{aligned} x - 3y + 9 &= 0 \\ xy - y^2 + 4 &= 0 \end{aligned}\right\}.$

10. $\left.\begin{aligned} x^2 + y^2 &= 100 \\ x + y &= 14 \end{aligned}\right\}.$

11. $\left.\begin{aligned} x^2 + y^2 &= 17 \\ 4x + y &= 15 \end{aligned}\right\}.$

12. $\left.\begin{aligned} 2x^2 - y^2 + 8 &= 0 \\ 3x - y - 2 &= 0 \end{aligned}\right\}.$

13. $\left.\begin{aligned} x^2 + xy &= 40 \\ 2x - 3y &= 1 \end{aligned}\right\}.$

14. $\left.\begin{aligned} x^2 - y^2 &= 13 \\ 3x - 2y &= 9 \end{aligned}\right\}.$

15. $\left.\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{5}{18} \\ xy &= 54 \end{aligned}\right\}.$

16. $\left.\begin{aligned} \frac{1}{x} - \frac{1}{y} &= \frac{1}{36} \\ x - 2y + 15 &= 0 \end{aligned}\right\}.$

17. $\left.\begin{aligned} x^2 + 4y + 11 &= 0 \\ 3x + 2y + 1 &= 0 \end{aligned}\right\}.$

18. $\left.\begin{aligned} x + 3y + 1 &= 0 \\ x + \frac{4y + 1}{x + 2y} &= 2(y + 1) \end{aligned}\right\}.$

19. $\left.\begin{aligned} x^2 + y^2 &= 106 \\ xy &= 45 \end{aligned}\right\}.$

20. $\left.\begin{aligned} x^2 + y^2 &= 52 \\ xy + 24 &= 0 \end{aligned}\right\}.$

21. $\left.\begin{aligned} x^2 - xy &= 3 \\ y^2 + xy &= 10 \end{aligned}\right\}.$

22. $\left.\begin{aligned} x^2 + xy + y^2 &= 37 \\ x^4 + x^2y^2 + y^4 &= 481 \end{aligned}\right\}.$

23. $\left.\begin{aligned} x^2 + 3xy + y^2 &= 1 \\ 3x^2 + xy + 3y^2 &= 13 \end{aligned}\right\}.$

24. $\left.\begin{aligned} 3xy + 2x + y &= 485 \\ 3x - 2y &= 0 \end{aligned}\right\}.$

25. $\left.\begin{aligned} x^2 - y^2 &= 0 \\ 3x^2 - 4xy + 5y^2 &= 9 \end{aligned}\right\}.$

26. $\left.\begin{aligned} xy + y^2 &= 4 \\ 2x^2 - y^2 &= 17 \end{aligned}\right\}.$

27. $\left.\begin{aligned} x^2 + 3xy &= 27 \\ xy - y^2 &= 2 \end{aligned}\right\}.$

28. $\left.\begin{aligned} x^2 + xy &= 60 \\ y^2 + xy &= 40 \end{aligned}\right\}.$

29. $\left.\begin{aligned} x^2 + 2xy - y^2 &= 28 \\ 3x^2 + 2xy + 2y^2 &= 72 \end{aligned}\right\}.$

30. $\left.\begin{aligned} x^2 - 4xy &= 45 \\ y^2 - xy &= 6 \end{aligned}\right\}.$

31. $\left.\begin{aligned} x^2 + 3xy &= 55 \\ 2y^2 + xy &= 18 \end{aligned}\right\}.$

32. $\left.\begin{aligned} x^2 - xy + y^2 &= 37 \\ x^2 + 2xy + 8 &= 0 \end{aligned}\right\}.$

33. $\left.\begin{aligned} x^2 + xy + 2y^2 &= 44 \\ 2x^2 - xy + y^2 &= 16 \end{aligned}\right\}.$

34. $\left.\begin{aligned} 8x^2 - 3xy - y^2 &= 40 \\ 9x^2 + xy + 2y^2 &= 60 \end{aligned}\right\}.$

35. $\left.\begin{aligned} 3x^2 + 3xy + y^2 &= 52 \\ 5x^2 + 7xy + 4y^2 &= 140 \end{aligned}\right\}.$

36. $\left.\begin{aligned} 4x^2 + 3xy + 5y^2 &= 27 \\ 7x^2 + 5xy + 9y^2 &= 47 \end{aligned}\right\}.$

37. $\left.\begin{aligned} 5x^2 + 3xy + 2y^2 &= 188 \\ x^2 - xy + y^2 &= 19 \end{aligned}\right\}.$

38. $\left.\begin{aligned} x^3 + y^3 &= 65 \\ x + y &= 5 \end{aligned}\right\}.$

39. $\left.\begin{aligned} x^3 - y^3 &= 98 \\ x - y &= 2 \end{aligned}\right\}.$

40. $\left.\begin{aligned} x^3 + y^3 &= 279 \\ x + y &= 3 \end{aligned}\right\}.$

41. $\left.\begin{aligned} x^3 - y^3 &= 218 \\ x - y &= 2 \end{aligned}\right\}.$

42. $\left.\begin{aligned} x^3 + y^3 &= 152 \\ x^2 - xy + y^2 &= 19 \end{aligned}\right\}.$

43. $\left.\begin{aligned} x^3 - y^3 &= 1304 \\ x^2 + xy + y^2 &= 163 \end{aligned}\right\}.$

44. $\left.\begin{aligned} x^3 + y^3 &= 91 \\ xy(x + y) &= 84 \end{aligned}\right\}.$

45. $\left.\begin{aligned} x^3 - y^3 &= 98 \\ x - y &= \frac{30}{xy} \end{aligned}\right\}.$

46. $\left.\begin{aligned} \frac{x^2}{y} + \frac{y^2}{x} &= \frac{27}{2} \\ \frac{1}{x} + \frac{1}{y} &= \frac{1}{2} \end{aligned}\right\}.$

47. $\left.\begin{aligned} \frac{x^2}{y}+\frac{y^2}{x}&=\frac{19}{6} \\ \frac{1}{x}+\frac{1}{y}&=\frac{1}{6} \end{aligned}\right\}.$

48. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}&=\frac{1}{2} \\ \frac{1}{x^2}+\frac{1}{y^2}&=\frac{5}{36} \end{aligned}\right\}.$

49. $\left.\begin{aligned} x^3-y^3&=7xy \\ x-y&=2 \end{aligned}\right\}.$

50. $\left.\begin{aligned} x^3+y^3&=\frac{27xy}{2} \\ x+y&=9 \end{aligned}\right\}.$

51. $\left.\begin{aligned} x^2+y^2&=\frac{5xy}{2} \\ x+y&=\frac{5xy}{6} \end{aligned}\right\}.$

52. $\left.\begin{aligned} x^2y^2-16xy+60&=0 \\ x+y&=7 \end{aligned}\right\}.$

53. $\left.\begin{aligned} x^2y^2&=4xy+12 \\ xy&=x+y+1 \end{aligned}\right\}.$

54. $\left.\begin{aligned} x^3+y^3&=\frac{35x^2y^2}{36} \\ x+y&=\frac{5xy}{6} \end{aligned}\right\}.$

55. $\left.\begin{aligned} x^2+y^2&=67-xy \\ x+y&=xy-5 \end{aligned}\right\}.$

56. $\left.\begin{aligned} x^3+y^3&=1-3xy \\ x^2+y^2&=xy+37 \end{aligned}\right\}.$

57. $\left.\begin{aligned} x^4+y^4&=706 \\ x+y&=2 \end{aligned}\right\}.$

58. $\left.\begin{aligned} x^5-y^5&=211 \\ x-y&=1 \end{aligned}\right\}.$

59. $\left.\begin{aligned} x^5+y^5&=3368 \\ x+y&=8 \end{aligned}\right\}.$

60. $\left.\begin{aligned} \frac{x^2}{y^2}+\frac{y^2}{x^2}&=17\left(\frac{xy}{16}\right)^2 \\ \frac{1}{x}+\frac{1}{y}&=\frac{3}{4} \end{aligned}\right\}.$

61. $\left.\begin{aligned} x^2+y^2&=xy+19 \\ x+y&=xy-7 \end{aligned}\right\}.$

62. $\left.\begin{aligned} \frac{x+y}{x-y}+\frac{x-y}{x+y}&=\frac{10}{3} \\ x^2+y^2&=45 \end{aligned}\right\}.$

63. $\left.\begin{aligned} x^4+x^2y^2+y^4&=133 \\ x^2-xy+y^2&=19 \end{aligned}\right\}.$

64. $\left.\begin{aligned} x^4+x^2y^2+y^4&=931 \\ x^2+xy+y^2&=49 \end{aligned}\right\}.$

65. $\left.\begin{aligned} x^2+xy+y^2&=84 \\ x+\sqrt{xy}+y&=6 \end{aligned}\right\}.$

66. $\left.\begin{aligned} x^2+y^2&=819-xy \\ x+y&=21+\sqrt{xy} \end{aligned}\right\}.$

67. $\left.\begin{aligned} x^4+y^4&=97 \\ x^2+y^2&=49-x^2y^2 \end{aligned}\right\}.$

68. $\left.\begin{aligned} 2x^2+3xy+12&=3y^2 \\ 3x+5y+1&=0 \end{aligned}\right\}.$

69. $\left.\begin{aligned} \frac{x}{a}+\frac{y}{b}&=1 \\ \frac{a}{x}+\frac{b}{y}&=4 \end{aligned}\right\}.$

70. $\left.\begin{aligned} x+y&=a \\ 4xy&=a^2-4b^2 \end{aligned}\right\}.$

71. $\left.\begin{aligned} x^2&=ax+by \\ y^2&=bx+ay \end{aligned}\right\}.$

72. $\left.\begin{aligned} x^2-xy&=a^2+b^2 \\ xy-y^2&=2ab \end{aligned}\right\}.$

73. $\left.\begin{aligned} x^2+y^2+x+y&=18 \\ xy&=6 \end{aligned}\right\}.$

74. $\left.\begin{aligned} x^4+y^4&=10(x^2+y^2)+72 \\ 2(x^2+y^2)&=5xy \end{aligned}\right\}.$

75. $\left.\begin{aligned} x^2+y^2&=2x^2y^2-15 \\ x+y&=xy+1 \end{aligned}\right\}.$

76. $\left.\begin{aligned} ay^2+bxy&=b \\ bx^2+axy&=a \end{aligned}\right\}.$

77. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}&=\frac{1}{x+y} \\ \frac{1}{x^2}+\frac{1}{y^2}&=\frac{1}{a^2} \end{aligned}\right\}.$

78. $\left.\begin{aligned} \frac{(x+y)^2}{a^2}+\frac{(x-y)^2}{b^2}&=8 \\ x^2+y^2&=2(a^2+b^2) \end{aligned}\right\}.$

79. $\left.\begin{aligned} x^2+y^2-8ab&=5(a^2+b^2) \\ xy-5ab&=2(a^2+b^2) \end{aligned}\right\}.$

80. $\left.\begin{aligned} x^2+y^2&=axy \\ x+y&=bxy \end{aligned}\right\}.$

81. $\left.\begin{aligned} 2(x^2+y^2)&=5xy-9ab \\ 2(a+b)(x+y)&=3(xy-ab) \end{aligned}\right\}.$

82. $\left.\begin{aligned} x^2+y^2+z^2&=49 \\ x+y+z&=11 \\ 2x+3y-4z&=6 \end{aligned}\right\}.$

83. $\left.\begin{aligned} xy+yz+xz&=40 \\ 4x=3y&=2z+4 \end{aligned}\right\}.$

84. $\left.\begin{aligned} x^2+y^2+z^2&=84 \\ x+y+z&=14 \\ y^2&=xz \end{aligned}\right\}.$

85. $\left.\begin{aligned} 2xy+x+y&=22 \\ 2yz+y+z&=58 \\ 2xz+x+z&=32 \end{aligned}\right\}.$

86. $\left.\begin{aligned} x^2+xy+xz&=a^2 \\ y^2+yz+xy&=2ab \\ z^2+xz+yz&=b^2 \end{aligned}\right\}.$

87. $\left.\begin{aligned} x^2+y^2&=24+5(x-y) \\ xy&=15 \end{aligned}\right\}.$

Exercise 28

1. If the length and breadth of a rectangle were each increased 1 foot, the area would be 48 square feet; if the length and breadth were each diminished 1 foot, the area would be 24 square feet. Find the length and the breadth of the rectangle.

2. A farmer laid out a rectangular lot containing 1200 square yards. He afterwards increased the width $1\frac{1}{3}$ yards and diminished the length 3 yards, thereby increasing the area by 60 square yards. Find the dimensions of the original lot.

3. The diagonal of a rectangle is 89 inches; if each side were 3 inches less, the diagonal would be 85 inches. Find the area of the rectangle.

4. The diagonal of a rectangle is 65 inches; if the rectangle were 3 inches shorter and 9 inches wider, the diagonal would still be 65 inches. Find the area of the rectangle.

5. The difference of two numbers is $\frac{3}{8}$ of the greater, and the sum of their squares is 356. Find the numbers.

6. The sum, the product, and the difference of the squares of two numbers are all equal. Find the numbers.

HINT. Represent the numbers by $x + y$ and $x - y$.

7. The sum of two numbers is 5, and the sum of their cubes is 65. Find the numbers.

8. The sum of two numbers is 11, and the cube of their sum exceeds the sum of their cubes by 792. Find the numbers.

9. A number is formed by two digits. The second digit is less by 8 than the square of the first digit; if 9 times the first digit is added to the number, the order of the digits is reversed. Find the number.

10. A number is formed by three digits, the third digit being the sum of the other two; the product of the first and third digits exceeds the square of the second by 5. If 396 is added to the number, the order of the digits is reversed. Find the number.

11. The numerator and denominator of a certain fraction are each greater by 1 than those of a second fraction; the sum of the two fractions is $\frac{17}{12}$. If the numerators were interchanged, the sum of the fractions would be $\frac{3}{2}$. Find the fractions.

12. There are two fractions. The numerator of the first is the square of the denominator of the second, and the numerator of the second is the square of the denominator of the first; the sum of the fractions is $\frac{35}{6}$, and the sum of their denominators 5. Find the fractions.

13. If the product of two numbers is increased by their sum, the result is 79. If their product is diminished by their sum, the result is 47. Find the numbers.

14. The sum of two numbers which are formed by the same two digits is $\frac{55}{18}$ of their difference; the difference of the squares of the numbers is 3960. Find the numbers.

15. The fore wheel of a carriage turns in a mile 132 times more than the hind wheel; if the circumference of each were increased 2 feet, the fore wheel would turn only 88 times more. Find the circumference of each wheel.

16. Two travelers, A and B, set out at the same moment from two distant towns, A to go from the first town to the second, and B from the second town to the first, and both travel at uniform rates. When they meet, A has traveled 30 miles farther than B. A finishes his journey 4 days, and B 9 days, after they meet. Find the distance between the towns, and the number of miles A and B each travel per day.

17. Two boys run in opposite directions around a rectangular field, the area of which is 1 acre; they start from one corner, and meet 13 yards from the opposite corner. One boy runs only $\frac{4}{5}$ as fast as the other. Find the length and breadth of the field.

18. A man walks from the base of a mountain to the summit, reaching the summit in $5\frac{1}{2}$ hours; during the last half of the distance he walks $\frac{1}{2}$ mile less per hour than during the first half. He descends in $3\frac{3}{4}$ hours, walking 1 mile per hour faster than during the first half of the ascent. Find the distance from the base to the summit and the rates of walking.

19. A garrison had bread for 11 days. If there had been 400 more men, each man's daily share would have been 2 ounces less; if there had been 600 less men, each man's daily share could have been increased by 2 ounces, and the bread would then have lasted 12 days. How many pounds of bread did the garrison have, and what was each man's daily share?

20. Three students, A, B, and C, agree to work out a set of problems in preparation for an examination; each is to do all the problems. A solves 9 problems per day and finishes the set 4 days before B; B solves 2 more problems per day than C, and finishes the set 6 days before C. Find the number of problems in the set.

21. A cistern can be filled by two pipes; one of these pipes can fill the cistern in 2 hours less time than the other; the cistern can be filled by both pipes running together in $1\frac{7}{8}$ hours. Find the time in which each pipe will fill the cistern.

22. A and B have a certain manuscript to copy between them. At A's rate of work he would copy the whole manuscript in 18 hours; B copies 9 pages per hour. A finishes his portion in as many hours as he copies pages per hour.

B is occupied with his portion 2 hours longer than A is with his. Find the number of pages copied by each.

23. A and B have 4800 circulars to stamp and intend to finish them in two days, 2400 each day. The first day A, working alone, stamps 800, and then A and B stamp the remaining 1600, A working in all 3 hours. The second day A works 3 hours and B 1 hour, and they accomplish only $\frac{9}{10}$ of their task for that day. Find the number of circulars each stamps per minute and the number of hours B works on the first day.

24. A, in running a race with B to a post and back, meets him 10 yards from the post. To come in even with A, B must increase his pace from this point $41\frac{3}{7}$ yards per minute. If, without changing his pace, he turns back on meeting A, he will come in 4 seconds behind A. Find the distance to the post.

25. A boat's crew, rowing at half their usual speed, row 3 miles down stream and back again, accomplishing the distance in 2 hours and 40 minutes. At full speed they can go over the same course in 1 hour and 4 minutes. Find the rate of the crew and of the current.

26. A farmer sold a number of sheep for \$286. He received for each sheep \$2 more than he paid for it, and gained thereby on the cost of the sheep $\frac{1}{2}$ as many per cent as each sheep cost dollars. Find the number of sheep.

27. A person has \$1300, which he divides into two parts and loans at different rates of interest in such a manner that the two portions produce equal returns. If the first portion had been loaned at the second rate of interest, it would have yielded annually \$36; if the second portion had been loaned at the first rate of interest, it would have yielded annually \$49. Find the two rates of interest.

28. A person has \$5000, which he divides into two portions and loans at different rates of interest in such a manner that the return from the first portion is double the return from the second portion. If the first portion had been loaned at the second rate of interest, it would have yielded annually \$245; if the second portion had been loaned at the first rate of interest, it would have yielded annually \$90. Find the two amounts and the two rates of interest.

29. A number is formed by three digits; 10 times the middle digit exceeds the square of half the sum of the three digits by 21; if 99 is added to the number, the digits are in reverse order; the number is 11 times the number formed by the first and third digits. Find the number.

30. A number is formed by three digits; the sum of the last two digits is the square of the first digit; the last digit is greater by 2 than the sum of the first and second; if 396 is added to the number, the digits are in reverse order. Find the number.

31. There are two numbers formed of the same two digits in reverse order. The sum of the numbers is 33 times the difference between the two digits, and the difference between the squares of the two numbers is 4752. Find the numbers.

32. A boat's crew, rowing at half their usual rate, row 2 miles down a river and back in 1 hour and 40 minutes. At their usual rate they would have gone over the same course in 40 minutes. Find the usual rate of the crew and the rate of the current.

33. A railroad train, after traveling 1 hour from A, meets with an accident which delays it 1 hour; it then proceeds at a rate 8 miles per hour less than its former rate and arrives at B 5 hours late. If the accident had happened 50 miles farther on, the train would have been only $3\frac{1}{3}$ hours late. Find the distance from A to B.

CHAPTER XII

EQUATIONS SOLVED AS QUADRATICS

187. An equation is in the *quadratic form* if it contains but two powers of the unknown, and if the exponent of one power is *twice* the exponent of the other power.

(1) Solve $8x^6 + 63x^3 = 8$.

This equation is in the quadratic form in x^3.

We have $$8x^6 + 63x^3 = 8.$$

Multiply by 32 and complete the square,

$$256x^6 + (\ \ ) + (63)^2 = 4225.$$

Extract the square root, $16x^3 + 63 = \pm 65.$

Hence, $$x^3 = \tfrac{1}{8} \text{ or } -8.$$

Extracting the cube root, we find two values of x to be $\frac{1}{2}$ and -2. To find the remaining roots, solve completely the two equations

$$x^3 = \tfrac{1}{8}, \qquad x^3 = -8.$$

We have $$8x^3 - 1 = 0,$$
or $$(2x-1)(4x^2+2x+1) = 0.$$
$$\therefore 2x - 1 = 0,$$
or $$4x^2 + 2x + 1 = 0.$$

Solving these, we find for three values of x,

$$\frac{1}{2},\ \frac{-1+\sqrt{-3}}{4},\ \frac{-1-\sqrt{-3}}{4}.$$

We have $$x^3 + 8 = 0,$$
or $$(x+2)(x^2-2x+4) = 0.$$
$$\therefore x + 2 = 0,$$
or $$x^2 - 2x + 4 = 0.$$

Solving these, we find for three values of x,

$$-2,\ 1+\sqrt{-3},\ 1-\sqrt{-3}.$$

These six values of x are the six roots of the given equation.

(2) Solve $\sqrt{x^3} - 3\sqrt[4]{x^3} = 40$.

Using fractional exponents, we have $x^{\frac{3}{2}} - 3x^{\frac{3}{4}} = 40$.

This equation is in the quadratic form in $x^{\frac{3}{4}}$, if we regard $x^{\frac{3}{4}}$ as the unknown number.

Complete the square, $4x^{\frac{3}{2}} - 12x^{\frac{3}{4}} + 9 = 169.$

Extract the root, $2x^{\frac{3}{4}} - 3 = \pm 13.$

$$\therefore 2x^{\frac{3}{4}} = 16 \text{ or } -10.$$

$$\therefore x^{\frac{3}{4}} - 8 = 0,$$

or

$$x^{\frac{3}{4}} + 5 = 0.$$

$$\therefore (x^{\frac{1}{4}} - 2)(x^{\frac{1}{2}} + 2x^{\frac{1}{4}} + 4) = 0; \text{ or } (x^{\frac{1}{4}} + 5^{\frac{1}{3}})(x^{\frac{1}{2}} - 5^{\frac{1}{3}}x^{\frac{1}{4}} + 5^{\frac{2}{3}}) = 0.$$

$$\therefore x^{\frac{1}{4}} = 2 \text{ or } -1 \pm \sqrt{-3}; \quad \text{or } x^{\frac{1}{4}} = -\sqrt[3]{5} \text{ or } \tfrac{1}{2}\sqrt[3]{5}(1 \pm \sqrt{-3}).$$

$$\therefore x = 16 \text{ or } 8(-1 \pm \sqrt{-3}); \text{ or } x = 5\sqrt[3]{5} \text{ or } \tfrac{5}{2}\sqrt[3]{5}(-1 \mp \sqrt{-3}).$$

Exercise 29

Solve:

1. $x^6 + 7x^3 = 8.$
2. $x^4 - 5x^2 + 4 = 0.$
3. $x^6 + 4x^3 = 96.$
4. $37x^2 - 9 = 4x^4.$
5. $16x^8 = 17x^4 - 1.$
6. $32x^{10} = 33x^5 - 1.$
7. $x^6 + 14x^3 + 24 = 0.$
8. $19x^4 + 216x^7 = x.$
9. $x^8 - 22x^4 + 21 = 0.$
10. $x^{2m} + 3x^m = 4.$
11. $x^{4n} - \dfrac{5x^{2n}}{3} = \dfrac{25}{12}.$
12. $x^{6n} + 3x^{3n} = 40.$
13. $x^{2m} + 2ax^m = 8a^2.$
14. $x^{-4} - 4x^{-2} = 12.$
15. $x^{-6} + 5x^{-3} - 36 = 0.$
16. $x^{-8} - 3x^{-4} - 154 = 0.$
17. $9x^{-4} + 4x^{-2} = 5.$
18. $4x^{\frac{1}{2}} - 3x^{\frac{1}{4}} = 10.$
19. $2x^{\frac{1}{3}} - 3x^{\frac{1}{6}} = 9.$
20. $\sqrt{x^5} = \sqrt[4]{x^5} + 12.$
21. $x = 9\sqrt{x} + 22.$
22. $\sqrt[3]{x^2} - 4\sqrt[3]{x} = 32.$
23. $2\sqrt{x^3} - 3\sqrt[4]{x^3} = 35.$
24. $\dfrac{1}{\sqrt[3]{x}} + \dfrac{1}{\sqrt[6]{x}} = \dfrac{3}{4}.$
25. $x^{-\frac{1}{2}} + x^{-\frac{1}{4}} = \frac{4}{9}.$
26. $3x^{-\frac{1}{3}} + 4x^{-\frac{1}{6}} = 20.$
27. $2x^{-\frac{2}{3}} - x^{-\frac{1}{3}} = 45.$
28. $4\sqrt[3]{x^{-2}} + 3\sqrt[3]{x^{-1}} = 27.$
29. $\sqrt[3]{2x} + \sqrt[3]{4x^2} = 72.$
30. $\sqrt{2x} + 4x = 1.$

188. Equivalent Equations. Two equations that involve the same unknown number are called *equivalent equations*, if the solutions of either include all the solutions of the other.

Thus, $7x - 3b = 5x + b$ and $4x = 8b$ are equivalent equations, for the solution of each is $x = 2b$.

A single equation is often equivalent to two or more equations.

Thus, the equation $x^3 + 1 = 0$ may be written

$$(x + 1)(x^2 - x + 1) = 0;$$

and this equation is equivalent to the two equations

$$x + 1 = 0 \text{ and } x^2 - x + 1 = 0.$$

In solving $x^3 + 1 = 0$, we should write it as $x + 1 = 0$ and $x^2 - x + 1 = 0$, and solve each of these equations.

If each member of an equation is multiplied by the same factor and this factor involves an unknown number of the equation, new solutions are in general introduced.

Thus, if we multiply $x - 3 = 0$ by $x - 5$, we get $(x - 3)(x - 5) = 0$, and introduce the solution of $x - 5 = 0$.

But if the multiplying factor is a denominator of a fraction of the equation, new solutions are in general not introduced.

Thus, $\dfrac{5}{x - 1} = 3 + x$ becomes, when multiplied by $x - 1$,

$$5 = (x - 1)(3 + x), \text{ or } x^2 + 2x - 8 = 0;$$

that is, $(x + 4)(x - 2) = 0$. Whence, $x = -4$ or 2.

Therefore, the solution $x = 1$ is not introduced, and this solution is the only solution that could be introduced by the factor $x - 1$.

In general, new solutions are not introduced in clearing an equation of fractions if we proceed as follows:

1. Combine fractions that have a common denominator.
2. Reduce fractions to their lowest terms.
3. Use the L.C.M. of the denominators for the multiplier.

If each member of an equation is raised to the same power, new solutions are, in general, introduced.

Thus, if we square each member of the equation $x = 2$, we have $x^2 = 4$, or $x^2 - 4 = 0$; that is, $(x + 2)(x - 2) = 0$.

Therefore, the solution of $x + 2 = 0$ was introduced by squaring both members of $x = 2$.

In solving an equation, if we raise each member to any power, we must reject the solutions of the resulting equation that do not satisfy the given equation.

Solve by clearing of radicals

$$\sqrt{x+4} + \sqrt{2x+6} = \sqrt{7x+14}.$$

Square, $\quad x + 4 + 2\sqrt{(x+4)(2x+6)} + 2x + 6 = 7x + 14.$

Transpose and combine, $\quad 2\sqrt{(x+4)(2x+6)} = 4x + 4.$

Divide by 2 and square, $\quad (x+4)(2x+6) = (2x+2)^2.$

Reduce, $\quad x^2 - 3x = 10.$

Therefore, $\quad x = 5$ or -2.

Of these two values only 5 will satisfy the given equation.

Squaring both numbers of the original equation is equivalent to transposing $\sqrt{7x+14}$ to the left member, and then multiplying by the rationalizing factor

$$\sqrt{x+4} + \sqrt{2x+6} + \sqrt{7x+14}.$$

The result reduces to

$$\sqrt{(x+4)(2x+6)} - (2x+2) = 0.$$

Transposing and squaring again is equivalent to multiplying by

$$(\sqrt{x+4} - \sqrt{2x+6} - \sqrt{7x+14})(\sqrt{x+4} - \sqrt{2x+6} + \sqrt{7x+14}).$$

Therefore, the equation $x^2 - 3x - 10 = 0$ is really obtained from

$$\begin{aligned}(\sqrt{x+4} &+ \sqrt{2x+6} - \sqrt{7x+14})\\ &\times (\sqrt{x+4} + \sqrt{2x+6} + \sqrt{7x+14})\\ &\times (\sqrt{x+4} - \sqrt{2x+6} - \sqrt{7x+14})\\ &\times (\sqrt{x+4} - \sqrt{2x+6} + \sqrt{7x+14}) = 0.\end{aligned}$$

This equation is satisfied by any value that will make any one of the *four* factors of its left member equal to zero. The first factor is 0 for $x = 5$, and the last factor is 0 for $x = -2$, while no value can be found to make the second or third factor vanish.

Since -2 does not satisfy the given equation but is introduced by multiplying by another equation, it is called an *extraneous value of x*.

189. Some radical equations may be solved as follows:

Solve $7x^2 - 5x + 8\sqrt{7x^2 - 5x + 1} = -8$.

Add 1 to each side,

$$7x^2 - 5x + 1 + 8\sqrt{7x^2 - 5x + 1} = -7.$$

Solving for $7x^2 - 5x + 1$, we have

$$7x^2 - 5x + 1 = 1, \text{ or } 7x^2 - 5x + 1 = 49.$$

Solving these, we find $0, \frac{5}{7}, 3, -\frac{16}{7}$ for the values of x.

All these values are extraneous values, and the given equation has no solution.

190. Various other equations may be solved by methods similar to that of the last section.

(1) Solve $x^4 - 4x^3 + 5x^2 - 2x - 20 = 0$.

Begin by attempting to extract the square root.

$$\begin{array}{r|l}
 & x^4 - 4x^3 + 5x^2 - 2x - 20\,(x^2 - 2x \\
 & x^4 \\
2x^2 - 2x & -4x^3 + 5x^2 \\
 & -4x^3 + 4x^2 \\ \hline
 & \qquad\qquad x^2 - 2x - 20
\end{array}$$

We see from the above that the equation may be written

$$(x^2 - 2x)^2 + (x^2 - 2x) - 20 = 0.$$

Solving, $\qquad x^2 - 2x = -5$, or $x^2 - 2x = 4$.

Solving these two equations, we find for the four values of x,

$$1 + 2\sqrt{-1}, \quad 1 - 2\sqrt{-1}, \quad 1 + \sqrt{5}, \quad 1 - \sqrt{5}.$$

(2) Solve $x^2 + \dfrac{1}{x^2} + x + \dfrac{1}{x} = 4$.

Add 2 to each member,

$$x^2 + 2 + \frac{1}{x^2} + x + \frac{1}{x} = 6,$$

or

$$\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 6.$$

Extract the root, $\qquad x + \dfrac{1}{x} = 2$, or $x + \dfrac{1}{x} = -3$.

Solving these two equations, we find for the four values of x,

$$1, \quad 1, \quad \frac{-3 + \sqrt{5}}{2}, \quad \frac{-3 - \sqrt{5}}{2}.$$

Exercise 30

Solve:

1. $\sqrt{x+4}+\sqrt{2x-1}=6.$
2. $\sqrt{13x-1}-\sqrt{2x-1}=5.$
3. $\sqrt{x}+\sqrt{4+x}=3.$
4. $\sqrt{x^2-9}+21=x^2.$
5. $\sqrt{x+1}+\sqrt{x+16}=\sqrt{x+25}.$
6. $\sqrt{2x+1}-\sqrt{x+4}=\dfrac{\sqrt{x-3}}{3}.$
7. $\sqrt{x+3}+\sqrt{x+8}=5\sqrt{x}.$
8. $\sqrt{x+7}+\sqrt{x-5}+\sqrt{3x+9}=0.$
9. $\sqrt{x+5}+\sqrt{8-2x}+\sqrt{9-4x}=0.$
10. $\sqrt{7-x}+\sqrt{3x+10}+\sqrt{x+3}=0.$
11. $\sqrt{2x^2+3x+7}=2x^2+3x-5.$
12. $x^2-3x+2=6\sqrt{x^2-3x-3}.$
13. $6x^2-3x-2=\sqrt{2x^2-x}.$
14. $15x-3x^2-16=4\sqrt{x^2-5x+5}.$
15. $6x^2-21x+20=\sqrt{4x^2-14x+16}.$
16. $\sqrt{36x^2+12x+33}=41-8x-24x^2.$
17. $4x^4-12x^3+5x^2+6x-15=0.$
18. $x^4-10x^3+35x^2-50x+24=0.$
19. $x^4-4x^3-10x^2+28x-15=0.$
20. $18x^4+24x^3-7x^2-10x-88=0.$
21. $4x^4-12x^3+17x^2-12x-12=0.$
22. $\sqrt{x}+\sqrt{x+3}=\dfrac{6}{\sqrt{x+3}}.$
23. $6+\sqrt{x^2-1}=\dfrac{16}{\sqrt{x^2-1}}.$

24. $\dfrac{1}{\sqrt{x+1}}+\dfrac{1}{\sqrt{x-1}}=\dfrac{1}{\sqrt{x^2-1}}.$

25. $\dfrac{\sqrt{x+2}-\sqrt{x-2}}{\sqrt{x+2}+\sqrt{x-2}}=\dfrac{x}{2}.$

26. $\dfrac{3x+\sqrt{4x-x^2}}{3x-\sqrt{4x-x^2}}=2.$

27. $\dfrac{\sqrt{3x^2+4}-\sqrt{2x^2+1}}{\sqrt{3x^2+4}+\sqrt{2x^2+1}}=\dfrac{1}{7}.$

28. $\dfrac{\sqrt{7x^2+4}+2\sqrt{3x-1}}{\sqrt{7x^2+4}-2\sqrt{3x-1}}=7.$

29. $\dfrac{\sqrt{5x-4}+\sqrt{5-x}}{\sqrt{5x-4}-\sqrt{5-x}}=\dfrac{2\sqrt{x}+1}{2\sqrt{x}-1}.$

30. $\sqrt{(x+a)^2+2ab+b^2}+x+a=b.$

31. $\dfrac{\sqrt{3}}{\sqrt{2x-1}-\sqrt{x-2}}=\dfrac{1}{\sqrt{x-1}}.$

32. $\sqrt{\dfrac{x}{4}+3}+\sqrt{\dfrac{x}{4}-3}=\sqrt{\dfrac{2x}{3}}.$

33. $\sqrt{1+\dfrac{x}{a}}-\sqrt{1-\dfrac{a}{x}}=1.$

34. $\sqrt{x^2+a^2+3ax}+\sqrt{x^2+a^2-3ax}=\sqrt{2a^2+2b^2}.$

35. $4x^{\frac{1}{2}}-3(x^{\frac{1}{2}}+1)(x^{\frac{1}{2}}-2)=x^{\frac{1}{2}}(10-3x^{\frac{1}{2}}).$

36. $(x^{\frac{2}{3}}-2)(x^{\frac{1}{3}}-4)=x^{\frac{1}{3}}(x^{\frac{1}{3}}-1)^2-12.$

37. $3\sqrt{x^6+17}+\sqrt{x^6+1}-2\sqrt{5x^3+41}=0.$

38. $\dfrac{1}{2}-\dfrac{3}{x}=\sqrt{\dfrac{1}{4}-\dfrac{1}{x}\sqrt{9-\dfrac{36}{x}}}.$

39. $\dfrac{2}{x+\sqrt{2-x^2}}+\dfrac{2}{x-\sqrt{2-x^2}}=x.$

40. $\dfrac{1}{1+\sqrt{1-x}}+\dfrac{1}{1-\sqrt{1-x}}=\dfrac{2x}{9}.$

41. $\dfrac{\sqrt{ax+b}+\sqrt{ax}}{\sqrt{ax+b}-\sqrt{ax}}=\dfrac{1+\sqrt{ax-b}}{1-\sqrt{ax-b}}.$

42. $\dfrac{\sqrt{a-x}+\sqrt{b-x}}{\sqrt{a-x}-\sqrt{b-x}}=\dfrac{\sqrt{x}+\sqrt{b}}{\sqrt{x}-\sqrt{b}}.$

43. $\sqrt{x}+\sqrt{a-\sqrt{ax+x^2}}=\sqrt{a}.$

44. $\left.\begin{aligned} x^2+y^2+x+y=48 \\ xy=12 \end{aligned}\right\}.$

45. $\left.\begin{aligned} x+y+\sqrt{x+y}=a \\ x-y+\sqrt{x-y}=b \end{aligned}\right\}.$

46. $\left.\begin{aligned} x^2+xy+y^2=a^2 \\ x+\sqrt{xy}+y=b \end{aligned}\right\}.$

47. $\left.\begin{aligned} \frac{3\sqrt{x}+2\sqrt{y}}{4\sqrt{x}-2\sqrt{y}}=6 \\ \frac{x^2+1}{16}=\frac{y^2-64}{x^2} \end{aligned}\right\}.$

48. $\left.\begin{aligned} \sqrt{x}-\sqrt{y}=x^{\frac{1}{4}}(\sqrt{x}+\sqrt{y}) \\ (x+y)^2=2(x-y)^2 \end{aligned}\right\}.$

49. $\left.\begin{aligned} \sqrt{\frac{3x}{x+y}}+\sqrt{\frac{x+y}{3x}}=2 \\ x+y=xy-54 \end{aligned}\right\}.$

CHAPTER XIII

PROPERTIES OF QUADRATIC EQUATIONS

191. If we represent the roots of the quadratic equation

$$ax^2 + bx + c = 0$$

by α and β, we have (§ 181)

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a};$$

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Adding, $$\alpha + \beta = -\frac{b}{a}.$$

Multiplying, $$\alpha\beta = \frac{c}{a}.$$

If we divide the equation $ax^2 + bx + c = 0$ through by a, we have the equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$; this may be written $x^2 + px + q = 0$, where $p = \frac{b}{a}$, $q = \frac{c}{a}$.

It appears, then, that if any quadratic equation is made to assume the form $x^2 + px + q = 0$, the following relations hold between the coefficients and roots of the equation:

1. The sum of the two roots is equal to the coefficient of x with its sign changed.
2. The product of the two roots is equal to the constant term.

Thus, the sum of the two roots of the equation $x^2 - 7x + 8 = 0$ is 7, and the product of the roots 8.

192. The expressions $\alpha + \beta$, $\alpha\beta$ are examples of **symmetric functions** of the roots. Any expression that involves both roots, and remains unchanged when the roots are interchanged, is a symmetric function of the roots.

From the relations $\alpha + \beta = -p$, $\alpha\beta = q$, the value of any symmetric function of the roots of a given quadratic may be found in terms of the coefficients.

Given that α and β are the roots of the quadratic $x^2 - 7x + 8 = 0$, we may find the values of symmetric functions of the roots as follows:

(1) $\alpha^2 + \beta^2$.

We have $\qquad \alpha + \beta = 7,$

and $\qquad \alpha\beta = 8.$

Square the first, $\qquad \alpha^2 + 2\alpha\beta + \beta^2 = 49$

Subtract, $\qquad 2\alpha\beta = 16$

and we have $\qquad \alpha^2 + \beta^2 = 33$

(2) $\alpha^3 + \beta^3$.

$$\alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 = 343$$

$3\alpha\beta(\alpha + \beta)$ or $\qquad 3\alpha^2\beta + 3\alpha\beta^2 = 168$

Subtract, $\qquad \alpha^3 + \beta^3 = 175$

(3) $\dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha}$.

This is $\qquad \dfrac{\alpha^3 + \beta^3}{\alpha\beta}$, which is $\dfrac{175}{8}$.

193. Resolution into Factors. By § 191, if α and β are the roots of the equation $x^2 + px + q = 0$, the equation may be written

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0.$$

The left member is the product of $x - \alpha$ and $x - \beta$, so that the equation may be also written

$$(x - \alpha)(x - \beta) = 0.$$

It appears, then, that the factors of the *quadratic expression* $x^2 + px + q$ are $x - \alpha$ and $x - \beta$, where α and β are the roots of the *quadratic equation* $x^2 + px + q = 0$.

The factors are real and different, real and alike, or imaginary, according as α and β are real and unequal, real and equal, or imaginary.

If $\beta = \alpha$, the equation becomes

$$(x - \alpha)(x - \alpha) = 0, \text{ or } (x - \alpha)^2 = 0.$$

If, then, the two roots of a quadratic equation are equal, the left member, when all the terms are transposed to that member, is a perfect square.

If the equation is in the form $ax^2 + bx + c = 0$, the left member may be written

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right), \text{ or } a(x - \alpha)(x - \beta). \qquad (\S\ 191)$$

194. If the roots of a quadratic equation are given, we can form the equation.

Form the equation of which the roots are 3 and $-\frac{5}{2}$.

The equation is $\quad (x - 3)(x + \frac{5}{2}) = 0,$

or $\quad (x - 3)(2x + 5) = 0,$

or $\quad 2x^2 - x - 15 = 0.$

195. **Quadratic expressions** may be factored by the principles of § 193.

(1) Resolve into two factors $x^2 - 5x + 3$.

Write the equation $x^2 - 5x + 3 = 0$.

The roots are found to be $\dfrac{5 + \sqrt{13}}{2}$ and $\dfrac{5 - \sqrt{13}}{2}$.

The factors of $x^2 - 5x + 3$ are

$$x - \frac{5 + \sqrt{13}}{2} \text{ and } x - \frac{5 - \sqrt{13}}{2}.$$

(2) Resolve into factors $3x^2 - 4x + 5$.

Write the equation $3x^2 - 4x + 5 = 0$.

The roots are found to be $\dfrac{2 + \sqrt{-11}}{3}$ and $\dfrac{2 - \sqrt{-11}}{3}$.

Therefore, the expression $3x^2 - 4x + 5$ may be written (§ 193)

$$3\left(x - \frac{2 + \sqrt{-11}}{3}\right)\left(x - \frac{2 - \sqrt{-11}}{3}\right).$$

Exercise 31

Form the equations of which the roots are:

1. $3,\ 2.$
2. $4,\ -5.$
3. $-6,\ -8.$
4. $\frac{2}{3},\ \frac{1}{2}.$
5. $-\frac{1}{3},\ -\frac{3}{4}.$
6. $a+3b,\ a-3b.$
7. $\dfrac{a+2b}{3},\ \dfrac{2a+b}{3}.$
8. $2+\sqrt{3},\ 2-\sqrt{3}.$
9. $-1+\sqrt{5},\ -1-\sqrt{5}.$
10. $1+\sqrt{\frac{2}{3}},\ 1-\sqrt{\frac{2}{3}}.$

Resolve into factors, real or imaginary:

11. $3x^2-15x-42.$
12. $9x^2-27x-70.$
13. $49x^2+49x+6.$
14. $169x^2-52x+4.$
15. $x^2-3x+4.$
16. $x^2+x+1.$
17. $4x^2-28x+49.$
18. $4x^2+12x+13.$

In Examples 19–27, α and β are to be taken as the roots of the equation $x^2-7x+8=0$.

Find the value of:

19. $(\alpha-\beta)^2.$
20. $\alpha^2\beta+\alpha\beta^2.$
21. $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}.$
22. $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}.$
23. $\dfrac{\alpha}{\beta^2}+\dfrac{\beta}{\alpha^2}.$
24. $\dfrac{\alpha^3+\beta^3}{\alpha+\beta}.$
25. $\dfrac{1}{\alpha^3}+\dfrac{1}{\beta^3}.$
26. $(\alpha^2-\beta^2)^2.$
27. $\dfrac{\alpha^3}{\beta^2}+\dfrac{\beta^3}{\alpha^2}.$

In Examples 28–33, α and β are to be taken as the roots of the equation $x^2+px+q=0$.

Find in terms of p and q the value of:

28. $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$.

29. $\alpha^2\beta+\alpha\beta^2$.

30. $\alpha^3+\beta^3$.

31. $\alpha^3\beta+\alpha\beta^3$.

32. $\alpha^4+\beta^4$.

33. $\dfrac{\alpha^2}{\beta^2}+\dfrac{\beta^2}{\alpha^2}$.

34. When will the roots of the equation $ax^2+bx+c=0$ be both positive? both negative? one positive and one negative?

196. The Roots in Special Cases. The values of the roots of the equation $ax^2+bx+c=0$ are (§ 191)

$$\frac{-b+\sqrt{b^2-4ac}}{2a},\quad \frac{-b-\sqrt{b^2-4ac}}{2a}. \qquad [1]$$

Multiplying both numerator and denominator of the first expression by $-b-\sqrt{b^2-4ac}$, and both numerator and denominator of the second expression by $-b+\sqrt{b^2-4ac}$, we obtain these new forms for the values of the roots:

$$\frac{2c}{-b-\sqrt{b^2-4ac}},\quad \frac{2c}{-b+\sqrt{b^2-4ac}}. \qquad [2]$$

We proceed to consider the following special cases:

1. Suppose a to be very small compared with b and c. In this case b^2-4ac differs but little from b^2, and its square root but little from b. The denominator of the first root in [2] will be very nearly $-2b$, and the root itself very nearly $-\dfrac{c}{b}$; the denominator of the second root in [2] will be very small, and the root itself numerically very large.

The smaller a is, the larger will the second root be, and the less will the first root differ from $-\dfrac{c}{b}$.

The first root may be found approximately by neglecting the x^2 term and solving the simple equation $bx + c = 0$. In fact, the quadratic equation itself approximates the form

$$0x^2 + bx + c = 0.$$

2. Suppose both a and b to be very small compared with c. In this case the first root, which differs but little from $-\frac{c}{b}$, also becomes very large, so that both roots are very large.

The smaller a and b are, the larger will the roots be. The quadratic equation in this case approximates the form

$$0x^2 + 0x + c = 0.$$

3. Suppose $c = 0$ while a and b are not zero. In this case the first root in [1] becomes zero, the second root becomes $-\frac{b}{a}$.

The quadratic equation becomes

$$ax^2 + bx = 0, \text{ or } x(ax + b) = 0;$$

one root is 0, the other is $-\frac{b}{a}$.

4. Suppose $b = 0$ and $c = 0$ while a is not zero. In this case the equation reduces to $ax^2 = 0$, of which both roots are zero.

5. Suppose $b = 0$ while a and c are not zero. In this case the two roots become $+\sqrt{-\frac{c}{a}}$ and $-\sqrt{-\frac{c}{a}}$.

The equation becomes the pure quadratic $ax^2 + c = 0$.

197. Collecting results, we have the following:

1. If a is very small compared with b and c; one root is very large.

2. If a and b are both very small compared with c; both roots are very large.

3. If $c = 0$, a and b not zero; one root is zero.

4. If $b = 0$, $c = 0$, a not zero; both roots are zero.

5. If $b = 0$, a and c not zero; the equation is a pure quadratic with roots numerically equal but opposite in sign.

198. Variable Coefficients. When the coefficients of an equation involve an undetermined number the character of the roots may depend on the value given to the unknown number.

For what values of m will the equation

$$2mx^2 + (5m + 2)x + (4m + 1) = 0$$

have its roots real and equal, real and unequal, imaginary?

We find $\quad b^2 - 4ac = (5m + 2)^2 - 8m(4m + 1)$

$$= 4 + 12m - 7m^2$$
$$= (2 - m)(2 + 7m).$$

Roots *equal*. In this case $b^2 - 4ac$ is zero. (§ 181)

$$\therefore 2 - m = 0, \text{ or } 2 + 7m = 0.$$
$$\therefore m = 2, \text{ or } m = -\tfrac{2}{7}.$$

Roots *real and unequal*. In this case $b^2 - 4ac$ is positive. (§ 181)

The factors $2 - m$, $2 + 7m$, are to be both positive or both negative.

If m lies between 2 and $-\frac{2}{7}$, both factors are positive; both factors cannot be negative.

Roots *imaginary*. In this case $b^2 - 4ac$ is negative. (§ 181)

Of the two factors $2 - m$, $2 + 7m$, one is positive, the other negative.

If m is greater than 2, $2 - m$ is negative and $2 + 7m$ positive; if m is less than $-\frac{2}{7}$, $2 + 7m$ is negative and $2 - m$ positive.

199. By a method similar to that of § 198 we can often obtain the **maximum** or the **minimum** value of a quadratic expression for real values of x.

(1) Find the maximum or the minimum value of $1 + x - x^2$ for real values of x.

Let $\quad 1 + x - x^2 = m.$

Solve, $\quad x = \dfrac{1 \pm \sqrt{5 - 4m}}{2}.$

Since x is real, we must have

$$5 > 4m \text{ or } 5 = 4m.$$

Therefore, $\quad 4m$ is not greater than 5.

That is, $\quad m$ is not greater than $\frac{5}{4}$.

The maximum value of $1 + x - x^2$ is $\frac{5}{4}$; for this value $x = \frac{1}{2}$.

(2) Find the minimum value of $x^2 + 3x + 4$ for real values of x.

Let $\qquad x^2 + 3x + 4 = m.$

Then, $\qquad x^2 + 3x = m - 4.$

Solve, $\qquad x = \dfrac{-3 \pm \sqrt{4m - 7}}{2}.$

Since x is real, we must have

$$4m > 7 \text{ or } 4m = 7.$$

Therefore, $\qquad 4m$ is not less than 7.

That is, $\qquad m$ is not less than $\frac{7}{4}$.

The minimum value of $x^2 + 3x + 4$ is $\frac{7}{4}$; for this value $x = -\frac{3}{2}$.

NOTE. Instead of solving for x, we might have used the condition for real roots, *viz.*, $b^2 - 4ac$ greater than or equal to zero.

200. The existence of a maximum or a minimum value may also be shown as follows:

Take the first expression of the last article,

$$1 + x - x^2.$$

This is $\qquad \frac{5}{4} - (\frac{1}{4} - x + x^2),$

or $\qquad \frac{5}{4} - (x - \frac{1}{2})^2.$

$(x - \frac{1}{2})^2$ is positive for all real values of x; its *least* value is zero, and in this case the given expression has its *greatest* value, $\frac{5}{4}$.

Similarly for any other expression.

Exercise 32

For what values of m are the two roots of each of the following equations equal, real and unequal, imaginary?

1. $(3m + 1)x^2 + 2(m + 1)x + m = 0.$
2. $(m - 2)x^2 + (m - 5)x + 2m - 5 = 0.$
3. $2mx^2 + x^2 - 6mx - 6x + 6m + 1 = 0.$
4. $mx^2 + 2x^2 + 2m - 3mx + 9x - 10 = 0.$
5. $6mx^2 + 8mx + 2m = 2x - x^2 - 1.$

For real values of x, find the maximum or the minimum value of each of the following expressions:

6. $x^2 - 6x + 13$.

7. $4x^2 - 12x + 16$.

8. $3 + 12x - 9x^2$.

9. $x^2 + 8x + 20$.

10. $4x^2 - 12x + 25$.

11. $25x^2 - 40x - 16$.

12. $\dfrac{x-6}{x^2}$.

13. $\dfrac{(x+12)(x-3)}{x^2}$.

14. $\dfrac{4x}{(x+2)^2}$.

15. $\dfrac{x^2 - x - 1}{x^2 - x + 1}$.

16. $\dfrac{x^2 + 2x - 3}{x^2 - 2x + 3}$.

17. $\dfrac{1}{2+x} - \dfrac{1}{2-x}$.

18. $\dfrac{x^2 + 3x + 5}{x^2 + 1}$.

19. $\dfrac{(x+1)^2}{x^2 - x + 1}$.

20. $\dfrac{2x^2 - 2x + 5}{x^2 - 2x + 3}$.

21. Divide a line $2a$ inches long into two parts such that the rectangle of these parts shall be the greatest possible.

22. Divide a line 20 inches long into two parts such that the hypotenuse of the right triangle of which the two parts are the legs shall be the least possible.

23. Divide $2a$ into two parts such that the sum of their square roots shall be a maximum.

24. Find the greatest rectangle that can be inscribed in a given triangle.

25. Find the greatest rectangle that can be inscribed in a given circle.

26. Find the rectangle of greatest perimeter that can be inscribed in a given circle.

CHAPTER XIV

SURDS AND IMAGINARIES

201. Quadratic Surds. *The product or the quotient of two dissimilar quadratic surds is a quadratic surd.*

For every quadratic surd, when simplified, has under the radical sign one or more factors raised only to the first power; and two surds which are *dissimilar* cannot have *all* these factors alike.

202. *The sum or the difference of two dissimilar quadratic surds cannot be a rational number, nor can it be expressed as a single surd.*

For, if $\sqrt{a} \pm \sqrt{b}$ could be equal to a rational number c, then squaring and transposing,

$$\pm 2\sqrt{ab} = c^2 - a - b.$$

Now, as the right side of this equation is rational, the left side should be rational; but $\sqrt{ab}$ cannot be rational (§ 201). Therefore, $\sqrt{a} \pm \sqrt{b}$ cannot be rational.

In like manner it may be shown that $\sqrt{a} \pm \sqrt{b}$ cannot be expressed as a single surd $\sqrt{c}$.

203. *A quadratic surd cannot be equal to the sum of a rational number and a surd.*

For, if $\sqrt{a}$ could be equal to $c + \sqrt{b}$, then squaring and transposing,

$$2c\sqrt{b} = a - b - c^2;$$

that is, a surd would be equal to a rational number; but this is impossible.

204. *If* a + $\sqrt{b} = x + \sqrt{y}$, *then* a *is equal to* x, *and* b *is equal to* y.

For, transposing, $\sqrt{b} - \sqrt{y} = x - a$; and if b were not equal to y, the difference of two unequal surds would be rational, which is impossible. (§ 202)

$$\therefore b = y, \text{ and } a = x.$$

In like manner, if $a - \sqrt{b} = x - \sqrt{y}$, a is equal to x, and b is equal to y.

An expression of the form $a + \sqrt{b}$, where $\sqrt{b}$ is a surd, is called a **binomial surd**.

205. Square Root of a Binomial Surd.

(1) Extract the square root of $a + \sqrt{b}$.

Let $\qquad \sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}.$

Square, $\qquad a + \sqrt{b} = x + 2\sqrt{xy} + y.$

$$\therefore x + y = a, \text{ and } 2\sqrt{xy} = \sqrt{b}. \qquad (§ 204)$$

From these two equations the values of x and y may be found.
Or, since $a = x + y$ and $\sqrt{b} = 2\sqrt{xy}$,

$$a - \sqrt{b} = x - 2\sqrt{xy} + y.$$

Extract the root, $\qquad \sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}.$

$$\therefore (\sqrt{a + \sqrt{b}})(\sqrt{a - \sqrt{b}}) = (\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}).$$

$$\therefore \sqrt{a^2 - b} = x - y.$$

And, as $\qquad a = x + y,$

the values of x and y may be found by addition and subtraction.

(2) Extract the square root of $7 + 4\sqrt{3}$.

Let $\qquad \sqrt{x} + \sqrt{y} = \sqrt{7 + 4\sqrt{3}}.$ [1]

Then, $\qquad \sqrt{x} - \sqrt{y} = \sqrt{7 - 4\sqrt{3}}.$ [2]

Multiply [1] by [2], $\qquad x - y = \sqrt{49 - 48}.$

$$\therefore x - y = 1.$$

But $\qquad x + y = 7.$

$$\therefore x = 4, \text{ and } y = 3.$$

$$\therefore \sqrt{x} + \sqrt{y} = 2 + \sqrt{3}.$$

$$\therefore \sqrt{7 + 4\sqrt{3}} = 2 + \sqrt{3}.$$

A root may often be obtained by inspection. For this purpose, write the given expression in the form $a + 2\sqrt{b}$, and determine the two numbers that have their sum equal to a, and their product equal to b.

(3) Find by inspection the square root of $18 + 2\sqrt{77}$.

The two numbers whose sum is 18 and product 77 are 11 and 7.

Then, $$18 + 2\sqrt{77} = 11 + 7 + 2\sqrt{11 \times 7}$$
$$= (\sqrt{11} + \sqrt{7})^2.$$

That is, $\sqrt{11} + \sqrt{7} =$ the square root of $18 + 2\sqrt{77}$.

(4) Find by inspection the square root of $75 - 12\sqrt{21}$.

It is necessary that the coefficient of the surd be 2; therefore,

$75 - 12\sqrt{21}$ must be put in the form $75 - 2\sqrt{756}$.

The two numbers whose sum is 75 and product 756 are 63 and 12.

Then, $$75 - 2\sqrt{756} = 63 + 12 - 2\sqrt{63 \times 12}$$
$$= (\sqrt{63} - \sqrt{12})^2.$$

That is, $\sqrt{63} - \sqrt{12} =$ the square root of $75 - 12\sqrt{21}$;

or $3\sqrt{7} - 2\sqrt{3} =$ the square root of $75 - 12\sqrt{21}$.

Exercise 33

Extract the square root of:

1. $14 + 6\sqrt{5}$.
2. $17 + 4\sqrt{15}$.
3. $10 + 2\sqrt{21}$.
4. $16 + 2\sqrt{55}$.
5. $9 - 2\sqrt{14}$.
6. $20 - 8\sqrt{6}$.
7. $9 - 6\sqrt{2}$.
8. $94 - 42\sqrt{5}$.
9. $13 - 2\sqrt{30}$.
10. $11 - 6\sqrt{2}$.
11. $14 - 4\sqrt{6}$.
12. $38 - 12\sqrt{10}$.
13. $103 - 12\sqrt{11}$.
14. $57 - 12\sqrt{15}$.
15. $3\frac{1}{2} - \sqrt{10}$.
16. $2a + 2\sqrt{a^2 - b^2}$.
17. $a^2 - 2b\sqrt{a^2 - b^2}$.
18. $87 - 12\sqrt{42}$.
19. $(a + b)^2 - 4(a - b)\sqrt{ab}$.

206. Orthotomic Numbers. The squares of *all* scalar numbers are positive scalar numbers; hence, a negative scalar number cannot be the square of a scalar number, and consequently the square root of a negative scalar number cannot be a scalar number (§ 144). For the complete treatment of evolution and of equations of the second and higher degrees, account must be taken of the square roots of negative scalar numbers, and as these roots are not scalar numbers it is necessary *to assume* a new series of numbers distinct from the scalar series, but such that the square of each and every number in the new series is a number in the negative branch of the scalar series. These new numbers being distinct from the scalar numbers require a distinguishing name, and accordingly they have been named **orthotomic numbers** or **imaginary numbers**. Hence,

An orthotomic number is any indicated square root of a negative scalar number or any scalar multiple thereof.

The complete series of orthotomic numbers includes a positive branch and a negative branch with zero as common origin.

207. Complex Numbers. The sum of any two scalar numbers is a scalar number, and it will presently be shown that the sum of any two orthotomic numbers is an orthotomic number, but the sum of a scalar number and an orthotomic number is evidently neither a scalar number nor an orthotomic number and therefore requires a distinctive name. The name generally given is **complex number**. Hence,

A complex number is the indicated sum or difference of a scalar number and an orthotomic number.

Thus, if g and h are scalar numbers either positive or negative but not zero, and p is a positive scalar number, but not zero, $h\sqrt{(-p)}$ is an orthotomic number and $g + h\sqrt{(-p)}$ is a complex number. If g and h may take any scalar values, *zero included*, the form $g + h\sqrt{(-p)}$ includes the whole assemblage of the scalar, the orthotomic, and the complex numbers.

Such assemblage is named the *uniplanar* or *coplanar assemblage of algebraic numbers.* It will be shown hereafter that this uniplanar assemblage includes *all* the numbers necessary to be considered in ordinary algebra; that is, the algebra of the four elementary operations, Addition, Subtraction, Multiplication, and Division, performed subject to the Laws of Uniformity, Association, Commutation, and Distribution as given in § 72.

208. The introduction of orthotomic numbers requires the *meanings* of the four elementary operations to be made more general in the algebra of complex numbers than they are in the algebra of scalar numbers, but these enlarged meanings must be consistent with the older meanings of scalar algebra and include them as special cases; and the elementary operations, when thus generalized, must be performed subject to the four fundamental laws which govern or define them in scalar algebra. (§ 34)

A full statement of these wider meanings with illustrative applications of them will be given in Chapter XXXIII.

209. It is necessary, however, to notice here the generalization of the Law of Signs which results from the action of the Associative and Commutative Laws of multiplication with the Law of Distribution of the square root operation over the factors of a product.

If a and b are both positive scalar numbers, the distribution of the square root operation over the factors of a product gives

$$+\sqrt{ab}=(+\sqrt{a})(+\sqrt{b})=(-\sqrt{a})(-\sqrt{b}), \qquad \text{(i)}$$

$$-\sqrt{ab}=(-\sqrt{a})(+\sqrt{b})=(+\sqrt{a})(-\sqrt{b}). \qquad \text{(ii)}$$

In extending this law to orthotomic numbers it is assumed that the law still holds when either factor (or both factors) under the radical sign is negative, provided the distribution is made over the factors taken with their signs unchanged. Thus,

$$+\sqrt{a(-b)}=(+\sqrt{a})(+\sqrt{-b})$$
$$=(-\sqrt{a})(-\sqrt{-b}), \qquad \text{(iii)}$$
$$-\sqrt{a(-b)}=(-\sqrt{a})(+\sqrt{-b})$$
$$=(+\sqrt{a})(-\sqrt{-b}); \qquad \text{(iv)}$$

and
$$+\sqrt{(-a)(-b)}=(+\sqrt{-a})(+\sqrt{-b})$$
$$=(-\sqrt{-a})(-\sqrt{-b}), \qquad \text{(v)}$$
$$-\sqrt{(-a)(-b)}=(-\sqrt{-a})(+\sqrt{-b})$$
$$=(+\sqrt{-a})(-\sqrt{-b}).^* \qquad \text{(vi)}$$

Hence, if $b=1$, we have as special cases of (iii) and (iv)

$$+\sqrt{-a}=+\sqrt{a(-1)}=(+\sqrt{a})(+\sqrt{-1}), \qquad \text{(vii)}$$

and
$$-\sqrt{-a}=-\sqrt{a(-1)}=(-\sqrt{a})(+\sqrt{-1}). \qquad \text{(viii)}$$

Now, by the Associative Law of multiplication, we have

$$+\sqrt{(-a)(-b)}=+\sqrt{(-1)\{a(-b)\}},$$

which, by (vii),
$$=(+\sqrt{-1})\{+\sqrt{a(-b)}\},$$

which, by the Associative and Commutative laws,

$$=(+\sqrt{-1})\{+\sqrt{(-1)(ab)}\},$$

which, by (vii),
$$=(+\sqrt{-1})(+\sqrt{-1})(+\sqrt{ab})$$
$$=(+\sqrt{-1})^2(+\sqrt{ab})$$
$$=-\sqrt{ab}.$$

But, by (v), $+\sqrt{(-a)(-b)}=(+\sqrt{-a})(+\sqrt{-b})$.

$$\therefore (+\sqrt{-a})(+\sqrt{-b})=-\sqrt{ab}.\dagger$$

* Notice that (iii), (iv), (v), and (vi) are all included in the forms (i) and (ii), if a and b are not restricted to be *positive* scalar numbers but may be any scalar numbers whatever. This generalization of (i) and (ii) is the proper statement of the distributive law of the square root.

† Notice that from this we have

$$(+\sqrt{-a})(+\sqrt{-b})=-\{(+\sqrt{a})(+\sqrt{b})\}.$$

Similarly, it may be shown that

$$(-\sqrt{-a})(-\sqrt{-b}) = -\sqrt{ab},$$
$$(-\sqrt{-a})(+\sqrt{-b}) = +\sqrt{ab},$$
$$(+\sqrt{-a})(-\sqrt{-b}) = +\sqrt{ab}.$$

The generalized Law of Signs in multiplication may now be enunciated as follows:

I. ***Two scalar factors with like signs give a positive scalar product; two scalar factors with unlike signs give a negative scalar product.***

II. ***Two orthotomic factors with like signs give a negative scalar product; two orthotomic factors with unlike signs give a positive scalar product.***

III. ***Two factors, the one scalar the other orthotomic, give a positive orthotomic product if the factors have like signs, a negative orthotomic product if the factors have unlike signs.***

210. The successive powers of $\sqrt{-1}$ are:

$$(\sqrt{-1})^2 = -1.$$
$$(\sqrt{-1})^3 = (\sqrt{-1})^2\sqrt{-1} \quad = (-1)\sqrt{-1} = -\sqrt{-1};$$
$$(\sqrt{-1})^4 = (\sqrt{-1})^2(\sqrt{-1})^2 = (-1)(-1) = +1;$$
$$(\sqrt{-1})^5 = (\sqrt{-1})^4\sqrt{-1} \quad = (+1)\sqrt{-1} = +\sqrt{-1}.$$

It appears that the successive powers of $\sqrt{-1}$ form the repeating series $+\sqrt{-1}$, -1, $-\sqrt{-1}$, $+1$; and so on.

211. Every orthotomic number is of the form $\pm m\sqrt{-p}$, wherein m and p are positive scalar numbers. Now,

$$\pm m\sqrt{-p} = \pm m\sqrt{p}\,\{\sqrt{-1}\},$$

and in this the factor $\pm m\sqrt{p}$ is a scalar number; hence, every orthotomic number may be written in the form $a\sqrt{-1}$, in which a is a scalar number; and, conversely, if a is a scalar

number, $a\sqrt{-1}$ will be an orthotomic number. Hence, the sum of two orthotomic numbers $a\sqrt{-1}$ and $b\sqrt{-1}$ is an orthotomic number or is zero, for

$$a\sqrt{-1} + b\sqrt{-1} = (a+b)\sqrt{-1},$$

and $(a+b)$ is a scalar number, a and b being scalar numbers, or is zero if $b = -a$.

212. Every imaginary number may be made to assume the form $a + b\sqrt{-1}$, where a and b are scalar numbers, and may be integers, fractions, or surds.

The form $a+b\sqrt{-1}$ is the **typical form of complex numbers.**

Reduce to the typical form $6 + \sqrt{-8}$.

This may be written $6 + \sqrt{8}\sqrt{-1}$, or $6 + 2\sqrt{2}\sqrt{-1}$; here $a = 6$, and $b = 2\sqrt{2}$.

213. The **algebraic sum** of two complex numbers is in general a complex number.

Add $a + b\sqrt{-1}$ and $c + d\sqrt{-1}$.

$$\begin{array}{r} a + b\sqrt{-1} \\ c + d\sqrt{-1} \\ \hline (a+c) + (b+d)\sqrt{-1} \end{array}$$

The sum is $(a+c) + (b+d)\sqrt{-1}$

This is a complex number unless $b + d = 0$, in which case the number is scalar, or $a + c = 0$, in which case the number is orthotomic.

214. The **product** of two complex numbers is in general a complex number.

Multiply $a + b\sqrt{-1}$ by $c + d\sqrt{-1}$.

$$\begin{array}{r} a + b\sqrt{-1} \\ c + d\sqrt{-1} \\ \hline ac + bc\sqrt{-1} \\ + ad\sqrt{-1} - bd \\ \hline (ac - bd) + (bc + ad)\sqrt{-1}, \end{array}$$

The product is $(ac - bd) + (bc + ad)\sqrt{-1}$,

which is a complex number unless $bc + ad = 0$ or $ac - bd = 0$.

215. The **quotient** of two complex numbers is in general a complex number.

Divide $a+b\sqrt{-1}$ by $c+d\sqrt{-1}$.

The quotient is $$\frac{a+b\sqrt{-1}}{c+d\sqrt{-1}}.$$

Multiply both numerator and denominator by $c-d\sqrt{-1}$.

Then, $$\frac{(a+b\sqrt{-1})(c-d\sqrt{-1})}{(c+d\sqrt{-1})(c-d\sqrt{-1})}$$
$$=\frac{(ac+bd)+(bc-ad)\sqrt{-1}}{c^2+d^2}$$
$$=\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}\sqrt{-1}.$$

This is a complex number in the typical form.

If $bc-ad=0$, the quotient is scalar.

216. Two expressions of the form $a+b\sqrt{-1}$, $a-b\sqrt{-1}$ are called **conjugate numbers**.

Add $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$.

The sum is $2a$.

Multiply $a+b\sqrt{-1}$ by $a-b\sqrt{-1}$.

$$\begin{array}{lr}
a+b\sqrt{-1} & \\
a-b\sqrt{-1} & \\
\hline
a^2+ab\sqrt{-1} & \\
\quad -ab\sqrt{-1} & +b^2 \\
\hline
a^2 & +b^2
\end{array}$$

The product is a^2+b^2

From the above it appears that the **sum** and the **product** of two conjugate numbers are both scalar.

The roots of a quadratic equation, if they are not scalar numbers, are conjugate numbers. (§ 181)

217. *A complex number cannot be equal to a scalar number.*

For, if possible, let $a+b\sqrt{-1}=c$.

Then, $b\sqrt{-1}=c-a$,

and $-b^2=(c-a)^2$.

Since b^2 and $(c-a)^2$ are both positive, we have a negative number equal to a positive number; but this is impossible.

218. *If two complex numbers are equal, the scalar parts are equal and the orthotomic parts are equal.*

For, let $a + b\sqrt{-1} = c + d\sqrt{-1}$.

Then, $(b - d)\sqrt{-1} = c - a$.

Square, $-(b - d)^2 = (c - a)^2$.

This equation is impossible unless $b = d$ and $a = c$.

219. *If* x *and* y *are scalar and* $\mathrm{x} + \mathrm{y}\sqrt{-1} = 0$, *then* $\mathrm{x} = 0$ *and* $\mathrm{y} = 0$.

For, $y\sqrt{-1} = -x$,

Square, $-y^2 = x^2$,

Transpose, $x^2 + y^2 = 0$,

This equation is true only when $x = 0$ and $y = 0$.

220. If the roots of $ax^2 + bx + c = 0$ are not scalar, then $ax^2 + bx + c$ is *positive* for all scalar values of x, if a is positive; and *negative* for all scalar values of x, if a is negative.

Let the two roots be $\gamma + \delta\sqrt{-1}$ and $\gamma - \delta\sqrt{-1}$, where γ and δ are scalar.

Then, by § 193, the expression $ax^2 + bx + c$ is identical with

$$a(x - \gamma - \delta\sqrt{-1})(x - \gamma + \delta\sqrt{-1}).$$

This product reduces to $a[(x - \gamma)^2 + \delta^2]$.

For all scalar values of x, $(x - \gamma)^2 + \delta^2$ is a positive scalar number. Hence, $ax^2 + bx + c$ is positive if a is positive, and negative if a is negative.

Examples:

(1) The roots of the equation $x^2 - 6x + 13 = 0$ are $3 + 2\sqrt{-1}$ and $3 - 2\sqrt{-1}$. The expression $x^2 - 6x + 13$ may be written $(x - 3)^2 + 4$, which is positive for all scalar values of x.

(2) The roots of the equation $12x - 13 - 4x^2 = 0$ are $\dfrac{3 + 2\sqrt{-1}}{2}$ and $\dfrac{3 - 2\sqrt{-1}}{2}$. The expression $12x - 13 - 4x^2$ may be written

$$-(4x^2 - 12x + 9 + 4) \text{ or } -[(2x - 3)^2 + 4],$$

which is negative for all scalar values of x.

The expressions $(x-3)^2+4$ and $-[(2x-3)^2+4]$ of Examples (1) and (2) cannot become zero for *any* scalar values of x; they accordingly have either a *minimum* value below which they cannot fall, or a *maximum* value above which they cannot rise. (§ 199)

Exercise 34

1. Multiply

$\sqrt{-8}$ by $\sqrt{-2}$; $2\sqrt{-3}$ by $4\sqrt{-27}$; $3\sqrt{-5}$ by $\dfrac{3}{\sqrt{27}}$.

2. Divide

$\sqrt{7}$ by $\sqrt{-3}$; $\sqrt{-8}$ by $\sqrt{-2}$; $3\sqrt{-6}$ by $\sqrt{2}\sqrt{-3}$.

3. Reduce to the typical form

$$4+\sqrt{-81};\ 5+2\sqrt{-6};\ (3+\sqrt{-27})^2.$$

Multiply:

4. $4+\sqrt{-3}$ by $4-\sqrt{-3}$.

5. $\sqrt{3}-2\sqrt{-2}$ by $\sqrt{3}+2\sqrt{-2}$.

6. $7+\sqrt{-27}$ by $4+\sqrt{-3}$.

7. $5+2\sqrt{-8}$ by $3-5\sqrt{-2}$.

8. $2\sqrt{3}-6\sqrt{-5}$ by $4\sqrt{3}-\sqrt{-5}$.

9. $\sqrt{a}+b\sqrt{-c}$ by $\sqrt{c}+a\sqrt{-b}$.

Divide:

10. 26 by $3+\sqrt{-4}$; 86 by $6-\sqrt{-7}$.

11. $3+\sqrt{-1}$ by $4+3\sqrt{-1}$.

12. $-9+19\sqrt{-2}$ by $3+\sqrt{-2}$.

Extract the square root of:

13. $1+4\sqrt{-3}$.

14. $10-8\sqrt{-6}$.

15. $-17+4\sqrt{-15}$.

16. $-38-15\sqrt{-28}$.

17. Show that $4x^2 - 12x + 25$ is positive for all scalar values of x, and find its minimum value.

18. Show that $6x - 4 - 9x^2$ is negative for all scalar values of x, and find its maximum value.

19. Show that each of the two complex roots of the equation $x^3 = 1$ is the square of the other complex root.

20. Show that, if ω is a complex root of $x^3 = 1$,

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z).$$

21. Find all the fourth roots of -1.

22. Find all the sixth roots of $+1$.

23. Find all the eighth roots of $+1$.

24. Reduce to the typical form

$$\frac{(2 - 3\sqrt{-1})(3 + 4\sqrt{-1})}{(6 + 4\sqrt{-1})(15 - 8\sqrt{-1})}.$$

Simplify:

25. $(1 + \sqrt{-5})^4 + (1 - \sqrt{-5})^4$.

26. $(1 + \sqrt{-2})^3 - (1 - \sqrt{-2})^3$.

27. $\sqrt{(3 + 4\sqrt{-1})} + \sqrt{(3 - 4\sqrt{-1})}$.

28. $\sqrt{(3 + 4\sqrt{-1})} - \sqrt{(3 - 4\sqrt{-1})}$.

29. $\sqrt{(5 + 2\sqrt{-6})} + \sqrt{(5 - 2\sqrt{-6})}$.

30. $\sqrt{(\sqrt{3} + \sqrt{-105})} - \sqrt{(\sqrt{3} - \sqrt{-105})}$.

CHAPTER XV

SIMPLE INDETERMINATE EQUATIONS

221. If a single equation involving two unknown numbers is given, and no other condition is imposed, the number of solutions of the equation is unlimited; for if one of the unknown numbers is assumed to have *any* particular value, a corresponding value of the other may be found.

Such an equation is called an **indeterminate equation.**

Although the number of solutions of an indeterminate equation is unlimited, the values of the unknown numbers are confined to a particular range; this range may be further limited by requiring that the unknown numbers shall be *positive integers.*

222. Every indeterminate equation of the first degree, in which x and y are the unknown numbers, may be made to assume the form

$$ax \pm by = \pm c,$$

where a, b, and c are positive integers and have no common factor.

223. The method of solving an indeterminate equation in positive integers is as follows:

(1) Solve $3x + 4y = 22$, in positive integers.

Transpose, $$3x = 22 - 4y.$$

$$\therefore x = 7 - y + \frac{1-y}{3},$$

the quotient being written as a mixed expression.

$$\therefore x + y - 7 = \frac{1-y}{3}.$$

Since the values of x and y are to be integral, $x + y - 7$ will be integral, and hence $\frac{1-y}{3}$ will be integral, though written in the *form* of a fraction.

Let $$\frac{1-y}{3} = m, \text{ an integer.}$$

Then, $$1 - y = 3m.$$
$$\therefore y = 1 - 3m.$$

Substitute this value of y in the original equation,

$$3x + 4 - 12m = 22.$$
$$\therefore x = 6 + 4m.$$

The equation $y = 1 - 3m$ shows that m may be 0, or have any negative integral value, but cannot have a positive integral value.

The equation $x = 6 + 4m$ further shows that m may be 0, but cannot have a negative integral value greater in absolute value than 1.

$$\therefore m \text{ may be } 0 \text{ or } -1,$$

and then $$\left.\begin{matrix} x = 6 \\ y = 1 \end{matrix}\right\}, \text{ or } \left.\begin{matrix} x = 2 \\ y = 4 \end{matrix}\right\}.$$

(2) Solve $5x - 14y = 11$, in positive integers.

Transpose, $$5x = 11 + 14y,$$
$$x = 2 + 2y + \frac{1+4y}{5}. \qquad [1]$$
$$\therefore x - 2y - 2 = \frac{1+4y}{5}.$$

Since x and y are to be integral, $x - 2y - 2$ will be integral, and hence $\frac{1+4y}{5}$ will be integral.

Let $$\frac{1+4y}{5} = m, \text{ an integer.}$$

Then, $$y = \frac{5m-1}{4},$$

or $$y = m + \frac{m-1}{4}. \qquad [2]$$

Now, $\frac{m-1}{4}$ must be integral.

Let $$\frac{m-1}{4} = n, \text{ an integer.}$$

Then, $$m = 4n + 1.$$

Substitute value of m in [2], $y = 5n + 1.$

Substitute value of y in [1], $x = 14n + 5.$

Obviously x and y will both be positive integers if n has *any* positive integral value.

Hence,
$$x = 5,\ 19,\ 33,\ 47,\ \cdots,$$
$$y = 1,\ \ 6,\ 11,\ 16,\ \cdots$$

Another method of solution is the following:

From the given equation we have $x = \dfrac{11 + 14y}{5}$.

Here y must be so taken that $11 + 14y$ is a multiple of 5; take $y = 1$, then $x = 5$, and we have *one* solution.

Now,
$$5x - 14y = 11,$$
and
$$5(5) - 14(1) = 11.$$
Subtract,
$$5(x - 5) - 14(y - 1) = 0,$$
or
$$\frac{x-5}{y-1} = \frac{14}{5}.$$

Since $x - 5$ and $y - 1$ are integers, $x - 5$ must be the same multiple of 14 that $y - 1$ is of 5.

Hence, if
$$x - 5 = 14m, \text{ then } y - 1 = 5m.$$
$$\therefore x = 14m + 5, \text{ and } y = 5m + 1.$$
Therefore,
$$x = 5,\ 19,\ 33,\ 47,\ \cdots,$$
and
$$y = 1,\ \ 6,\ 11,\ 16,\ \cdots$$

It will be seen from [1] and [2] that when only positive integers are required the number of solutions will be *limited* or *unlimited* according as the sign connecting x and y is *positive* or *negative*.

(3) Find the least number that when divided by 14 and 5 will give remainders 1 and 3 respectively.

If N represents the number, then
$$\frac{N-1}{14} = x, \text{ and } \frac{N-3}{5} = y.$$
$$\therefore N = 14x + 1, \text{ and } N = 5y + 3.$$
$$\therefore 14x + 1 = 5y + 3.$$
$$5y = 14x - 2,$$
$$5y = 15x - 2 - x.$$
$$\therefore y = 3x - \frac{2+x}{5}.$$
Let
$$\frac{2+x}{5} = m, \text{ an integer.}$$
$$\therefore x = 5m - 2.$$

$$y = \tfrac{1}{5}(14x - 2), \text{ from original equation.}$$

$$\therefore y = 14m - 6.$$

If $m = 1$, $\quad x = 3$, and $y = 8$,

$$\therefore N = 14x + 1 = 5y + 3 = 43.$$

(4) Solve $5x + 6y = 30$, so that x may be a multiple of y, and both x and y positive.

Let $\quad x = my.$

Then, $\quad (5m + 6)y = 30.$

$$\therefore y = \frac{30}{5m + 6},$$

and
$$x = \frac{30m}{5m + 6}.$$

If $m = 2$, $\quad x = 3\tfrac{3}{4}$, $y = 1\tfrac{7}{8}$.

If $m = 3$, $\quad x = 4\tfrac{2}{7}$, $y = 1\tfrac{3}{7}$;

and so on.

(5) Solve $14x + 22y = 71$, in positive integers.

$$x = 5 - y + \frac{1 - 8y}{14}.$$

If we multiply the fraction by 7 and reduce, the result is $-4y + \frac{1}{2}$, a form which shows that there can be no *integral* solution.

There can be no integral solution of $ax \pm by = \pm c$ if a and b have a common factor not common also to c; for, if d is a factor of a and also of b, but not of c, the equation may be written

$$mdx \pm ndy = \pm c, \text{ or } mx \pm ny = \pm \frac{c}{d};$$

which is impossible, since $\frac{c}{d}$ is a fraction, and $mx \pm ny$ is an integer, if x and y are integers.

Exercise 35

Solve in positive integers:

1. $x + y = 12.$
2. $2x + 11y = 83.$
3. $4x + 9y = 53.$
4. $8x + 5y = 74.$
5. $5x + 3y = 105.$
6. $\frac{3}{4}x + 5y = 92.$
7. $\frac{3}{4}x + \frac{1}{4}y = 27.$
8. $\frac{2}{3}x + \frac{3}{4}y = 53.$

Solve in least possible integers:

9. $7x - 2y = 12$.
10. $9x - 5y = 21$.
11. $7x - 4y = 45$.
12. $11x - 5y = 73$.
13. $15x - 47y = 11$.
14. $23x - 14y = 99$.

15. Find two numbers which, multiplied respectively by 7 and 17, have for the sum of their products 1135.

16. If two numbers are multiplied respectively by 8 and 17, the difference of their products is 10. What are the numbers?

17. If two numbers are multiplied respectively by 7 and 15, the first product is greater by 12 than the second. Find the numbers.

18. Divide 89 in two parts, one of which is divisible by 3, and the other by 8.

19. Divide 314 in two parts, one of which is a multiple of 11, and the other a multiple of 13.

20. What is the smallest number which, divided by 5 and by 7, gives each time 4 for a remainder?

21. The difference between two numbers is 151. The first divided by 8 has 5 for a remainder, and 4 must be added to the second to make it divisible by 11. What are the numbers?

22. Find pairs of fractions whose denominators are 24 and 16, and whose sum is $\frac{19}{24}$.

23. How can one pay a sum of \$87, giving only bills of \$5 and \$2?

24. A man buys calves at \$5 apiece, and pigs at \$3 apiece. He spends in all \$114. How many did he buy of each?

25. A person bought 40 animals, consisting of pigs, geese, and chickens, for \$40. The pigs cost \$5 apiece, the geese \$1, and the chickens 25 cents each. Find the number he bought of each.

26. Solve $18x - 5y = 70$ so that y may be a multiple of x, and both positive.

27. Solve $8x + 12y = 23$ so that x and y may be positive, and their sum an integer.

28. Divide 70 into three parts which shall give integral quotients when divided by 6, 7, 8 respectively and the sum of the quotients shall be 10.

29. In how many ways can \$3.60 be paid with dollars and twenty-cent pieces?

30. In how many ways can 300 pounds be weighed with 7 and 9 pound weights?

31. Find the general form of the numbers that, divided by 2, 3, 7, have for remainders 1, 2, 5 respectively.

32. Find the general form of the numbers that, divided by 7, 8, 9, have for remainders 6, 7, 8 respectively.

33. A farmer buys oxen, sheep, and hens. The whole number bought is 100, and the total cost £100. If the oxen cost £5, the sheep £1, and the hens 1*s.* each, how many of each does he buy?

34. A farmer sells 15 calves, 14 lambs, and 13 pigs, and receives \$200. Some days after, at the same price, he sells 7 calves, 11 lambs, and 16 pigs, for which he receives \$141. What is the price of each?

CHAPTER XVI

INEQUALITIES

224. An **inequality** is a statement that two expressions do not have the same value; that is, a statement that two expressions do not represent the same number.

Every inequality consists of two expressions connected by a sign of inequality; the two expressions are called the **sides** or **members** of the inequality.

225. We say that $a > b$ when $a - b$ is *positive;* that $a < b$ when $a - b$ is *negative.*

226. The symbols $\nless$ and $\ngtr$ are used for the words *not less than* and *not greater than* respectively.

227. In working with inequalities the following principles are easily shown to be true:

The sign of an inequality remains unchanged if both members are increased or diminished by the same number; if both members are multiplied or divided by the same positive number; if both members are raised to any odd power, or to any power when both members are positive.

The sign of an inequality is reversed if both members are multiplied or divided by the same negative number; if both members are raised to the same even power when both members are negative.

228. Fundamental Theorem. *If* a *and* b *are unequal scalar numbers,* $a^2 + b^2 > 2ab$.

For $(a - b)^2$ must be positive.

That is, $$a^2 - 2ab + b^2 > 0.$$

$$\therefore a^2 + b^2 > 2ab. \qquad (\S\ 227)$$

(1) If a and b are unequal positive scalar numbers, show that $a^3 + b^3 > a^2b + ab^2$.

We shall have $a^3 + b^3 > a^2b + ab^2$,
if (dividing by $a + b$) $a^2 - ab + b^2 > ab$,
if $a^2 + b^2 > 2ab$.
But $a^2 + b^2 > 2ab$. (§ 228)
$\therefore a^3 + b^3 > a^2b + ab^2$.

(2) Show that $a^2 + b^2 + c^2 > ab + ac + bc$.

Now, $a^2 + b^2 > 2ab$,
$a^2 + c^2 > 2ac$, (§ 228)
$b^2 + c^2 > 2bc$.
Add, $2a^2 + 2b^2 + 2c^2 > 2ab + 2ac + 2bc$.
$\therefore a^2 + b^2 + c^2 > ab + ac + bc$.

Exercise 36

Show that, the letters being unequal positive scalar numbers:

1. $a^2 + 3b^2 > 2b(a + b)$.
2. $a^3b + ab^3 > 2a^2b^2$.
3. $(a^2 + b^2)(a^4 + b^4) > (a^3 + b^3)^2$.
4. $a^2b + a^2c + ab^2 + b^2c + ac^2 + bc^2 > 6abc$.
5. The sum of any fraction and its reciprocal > 2.
6. If $x^2 = a^2 + b^2$, and $y^2 = c^2 + d^2$, $xy \nless ac + bd$, or $ad + bc$.
7. $ab + ac + bc < (a + b - c)^2 + (a + c - b)^2 + (b + c - a)^2$.
8. Which is the greater, $(a^2 + b^2)(c^2 + d^2)$ or $(ac + bd)^2$?
9. Which is the greater, $a^4 - b^4$ or $4a^3(a - b)$ when $a > b$?
10. Which is the greater, $\sqrt{\dfrac{a^2}{b}} + \sqrt{\dfrac{b^2}{a}}$ or $\sqrt{a} + \sqrt{b}$?
11. Which is the greater, $\dfrac{a + b}{2}$ or $\dfrac{2ab}{a + b}$?
12. Which is the greater, $\dfrac{a}{b^2} + \dfrac{b}{a^2}$ or $\dfrac{1}{b} + \dfrac{1}{a}$?

CHAPTER XVII

RATIO, PROPORTION, AND VARIATION

229. Ratio of Numbers. The relative magnitude of two numbers is called their **ratio**, when expressed by the indicated quotient of the first by the second.

Thus, the ratio of a to b is $\frac{a}{b}$, or $a \div b$, or $a : b$; the quotient is generally written in the last form when it is intended to express a ratio.

The first term of a ratio is called the **antecedent**, and the second term the **consequent**.

When the antecedent and consequent are interchanged the resulting ratio is called the **inverse** of the given ratio.

Thus, the ratio $3:6$ is the *inverse* of the ratio $6:3$.

230. A ratio is not changed if both its terms are multiplied by the same number.

Thus, the ratio $a:b$ is represented by $\frac{a}{b}$, the ratio $ma:mb$ is represented by $\frac{ma}{mb}$; and since $\frac{ma}{mb} = \frac{a}{b}$, we have $ma:mb = a:b$.

A ratio is changed if its terms are multiplied by different multipliers.

If $$m \neq n,$$
then $$ma \neq na,$$
and $$\frac{ma}{nb} \neq \frac{na}{nb}.$$
But $$\frac{na}{nb} = \frac{a}{b}.$$
$$\therefore \frac{ma}{nb} \neq \frac{a}{b},$$
or $$ma : nb \neq a : b.$$

231. Ratios are **compounded** by taking the product of the fractions that represent them.

Thus, the ratio compounded of $a:b$ and $c:d$ is $ac:bd$.

The ratio compounded of $a:b$ and $a:b$ is called the **duplicate** ratio $a^2:b^2$.

The ratio compounded of $a:b$, $a:b$, and $a:b$ is called the **triplicate** ratio $a^3:b^3$; and so on.

232. Ratios are **compared** by comparing the fractions that represent them.

Thus, $$a:b > \text{ or } < c:d$$

according as $$\frac{a}{b} > \text{ or } < \frac{c}{d}.$$

233. Proportion of Numbers. Four numbers, a, b, c, d, are in **proportion** when the ratio $a:b$ is equal to the ratio $c:d$.

We then write $a:b=c:d$ (read, *the ratio of* a *to* b *equals the ratio of* c *to* d, or a *is to* b *as* c *is to* d).

A proportion is also written $a:b::c:d$.

The four numbers, a, b, c, d, are called **proportionals**; a and d are called the **extremes**, b and c the **means**.

234. The **fourth proportional** to three given numbers is the fourth term of the proportion which has for its first three terms the three given numbers *taken in order*.

Thus, d is the fourth proportional to a, b, and c in the proportion
$$a:b=c:d.$$

235. The numbers a, b, c, d, e are said to be in **continued proportion** if $a:b=b:c=c:d=d:e$.

If three numbers are in continued proportion, the second is called the **mean proportional** between the other two numbers, and the third is called the **third proportional** to the other two numbers.

Thus, b is the mean proportional between a and c in the proportion $a:b=b:c$; and c is the third proportional to a and b.

236. *If four numbers are in proportion, the product of the extremes is equal to the product of the means.*

For, if $a : b = c : d$,

then $\dfrac{a}{b} = \dfrac{c}{d}$.

Multiply by bd, $ad = bc$.

The equation $ad = bc$ gives

$$a = \frac{bc}{d}, \qquad b = \frac{ad}{c};$$

so that an extreme may be found by dividing the product of the means by the other extreme; and a mean may be found by dividing the product of the extremes by the other mean. If three terms of a proportion are given, it appears from the above that the fourth term has one, and but one, value.

237. *If the product of two numbers is equal to the product of two others, either two may be made the extremes of a proportion and the other two the means.*

For, if $ad = bc$,

divide by bd, $\dfrac{ad}{bd} = \dfrac{bc}{bd}$,

or $\dfrac{a}{b} = \dfrac{c}{d}$.

$$\therefore a : b = c : d.$$

238. *If four numbers,* a, b, c, d, *are in proportion, they are in proportion by* **inversion**; *that is,* b *is to* a *as* d *is to* c.

For, if $a : b = c : d$,

then $\dfrac{a}{b} = \dfrac{c}{d}$,

and $1 \div \dfrac{a}{b} = 1 \div \dfrac{c}{d}$,

or $\dfrac{b}{a} = \dfrac{d}{c}$.

$$\therefore b : a = d : c.$$

239. *If four numbers,* a, b, c, d, *are in proportion, they are in proportion by* **composition**; *that is,* $a + b : b = c + d : d$.

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d},$$

and $$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

or $$\frac{a+b}{b} = \frac{c+d}{d}.$$

$$\therefore a + b : b = c + d : d.$$

240. *If four numbers,* a, b, c, d, *are in proportion, they are in proportion by* **division**; *that is,* $a - b : b = c - d : d$.

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d},$$

and $$\frac{a}{b} - 1 = \frac{c}{d} - 1,$$

or $$\frac{a-b}{b} = \frac{c-d}{d}.$$

$$\therefore a - b : b = c - d : d.$$

241. *If four numbers,* a, b, c, d, *are in proportion, they are in proportion by* **composition and division**; *that is,*

$$a + b : a - b = c + d : c - d.$$

For, from § 239, $$\frac{a+b}{b} = \frac{c+d}{d},$$

and from § 240, $$\frac{a-b}{b} = \frac{c-d}{d}.$$

Divide, $$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

$$\therefore a + b : a - b = c + d : c - d.$$

242. *If four numbers,* a, b, c, d, *are in proportion, they are in proportion by* **alternation**; *that is,* $a : c = b : d$.

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d}.$$

Multiply by $\frac{b}{c}$,
$$\frac{ab}{bc}=\frac{bc}{cd},$$
or
$$\frac{a}{c}=\frac{b}{d}.$$
$$\therefore\ a:c=b:d.$$

243. *Like powers of the terms of a proportion are in proportion.*

For, if
$$a:b=c:d,$$
then
$$\frac{a}{b}=\frac{c}{d}.$$

Raise both sides to the nth power,
$$\frac{a^n}{b^n}=\frac{c^n}{d^n}.$$
$$\therefore\ a^n:b^n=c^n:d^n.$$

244. If $a:b=c:d$, any ratio whose terms are two polynomials in a and b, homogeneous and both of the same degree, is equal to the ratio whose terms are found from those of the preceding ratio by substituting c for a and d for b.

To prove this in any particular case, it will be found sufficient to substitute ra for b and rc for d.

245. *In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

For, if
$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h},$$
we may put r for each of these ratios.

Then,
$$\frac{a}{b}=r,\quad \frac{c}{d}=r,\quad \frac{e}{f}=r,\quad \frac{g}{h}=r.$$
$$\therefore\ a=br,\quad c=dr,\quad e=fr,\quad g=hr.$$
$$\therefore\ a+c+e+g=(b+d+f+h)r.$$
$$\therefore\ \frac{a+c+e+g}{b+d+f+h}=r=\frac{a}{b}.$$
$$\therefore\ a+c+e+g:b+d+f+h=a:b.$$

In like manner, it may be shown that
$$ma+nc+pe+qg:mb+nd+pf+qh=a:b.$$

246. *If four numbers,* a, b, c, d, *are in* **continued proportion,** *then* $a:c=a^2:b^2$ *and* $a:d=a^3:b^3$.

For, if $$\frac{a}{b}=\frac{b}{c}=\frac{c}{d},$$

then $$\frac{a}{b}\times\frac{b}{c}=\frac{a}{b}\times\frac{a}{b},$$

or $$\frac{a}{c}=\frac{a^2}{b^2}.$$

$$\therefore\ a:c=a^2:b^2.$$

Also, $$\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b},$$

or $$\frac{a}{d}=\frac{a^3}{b^3}.$$

$$\therefore\ a:d=a^3:b^3.$$

247. *The mean proportional between two numbers is equal to the square root of their product.*

For, if $$a:b=b:c,$$

then $$\frac{a}{b}=\frac{b}{c},$$

and $$b^2=ac.$$

$$\therefore\ b=\sqrt{ac}.$$

248. *The products of the corresponding terms of two or more proportions are in proportion.*

For, if $$a:b=c:d,$$
$$e:f=g:h,$$
$$k:l=m:n,$$

then $$\frac{a}{b}=\frac{c}{d},\quad \frac{e}{f}=\frac{g}{h},\quad \frac{k}{l}=\frac{m}{n}.$$

Take the product of the left members, and also of the right members of these equations,

$$\frac{aek}{bfl}=\frac{cgm}{dhn}.$$

$$\therefore\ aek:bfl=cgm:dhn.$$

249. The laws that have been established for ratios should be remembered when ratios are expressed in fractional form.

[illegible] Solve $\frac{x^2 - x - 1}{x^2 - x - 1} = \frac{x^2 - x - 2}{x^2 - x - 2}$. [illegible]

By composition and division,

$$\frac{2x^2}{2x - [illegible]} = \frac{2x^2}{-2x - 2)}.$$

This equation is satisfied when $x = 0$

or, dividing by $\frac{2x^2}{2}$ when $\frac{1}{x - 1} = \frac{1}{2 - x}$,

that is, when $x = \frac{3}{2}$.

[illegible] If $a : b = c : d$, show that

$$a^2 + ab : b^2 - ab = c^2 + cd : d^2 - cd.$$

If $$\frac{a}{b} = \frac{c}{d},$$

then $$\frac{a + b}{a - b} = \frac{c + d}{c - d}. \qquad (\S\ 241)$$

and $$\frac{a}{-b} = \frac{c}{-d}.$$

$$\therefore \frac{a}{-b} \times \frac{a + b}{a - b} = \frac{c}{-d} \times \frac{c + d}{c - d}; \qquad (\S\ 248)$$

that is, $$\frac{a^2 + ab}{b^2 - ab} = \frac{c^2 + cd}{d^2 - cd}.$$

or $$a^2 + ab : b^2 - ab = c^2 + cd : d^2 - cd.$$

Exercise 37

1. Write the ratio compounded of $3:5$ and $8:7$. Which of these ratios is increased, and which is diminished by the compounding?

2. Compound the duplicate ratio of $4:15$ with the triplicate of $5:2$.

3. Arrange in order of magnitude the ratios $3:4$, $23:25$, $10:11$.

Find the ratio compounded of:

4. $3:5$, $10:21$, $14:15$.

5. $7:9$, $102:105$, $15:17$.

6. $a^2 - x^2 : a^2 + 3ax + 2x^2$ and $a + x : a - x$.

7. $x^2 - 4 : 2x^2 - 5x + 3$ and $x - 1 : x - 2$.

8. Show that the ratio $a : b$ is the duplicate of the ratio

$$a + c : b + c \text{ if } c^2 = ab.$$

9. Two numbers are in the ratio $2 : 5$, and if 6 is added to each, the sums are in the ratio $4 : 7$. Find the numbers.

10. What must be added to each of the terms of the ratio $m : n$ that it may become equal to the ratio $p : q$?

11. If x and y are such that, when they are added to the antecedent and consequent respectively of the ratio $a : b$, its value is unaltered, show that $x : y = a : b$.

Find x from the proportions:

12. $27 : 90 = 45 : x$.

13. $11\frac{1}{4} : 4\frac{1}{2} = 3\frac{3}{4} : x$.

14. $\dfrac{3a}{5b} : \dfrac{12a}{7c} = \dfrac{14c}{15b} : x$.

Find the third proportional to:

15. $\frac{27}{16}$ and $\frac{5}{12}$.

16. $\dfrac{a^2 - b^2}{c}$ and $\dfrac{a - b}{c}$.

Find the mean proportional between:

17. 3 and $16\frac{1}{3}$.

18. $\dfrac{(m-5)^2}{m+5}$ and $\dfrac{(m+5)^2}{m-5}$.

If $a : b = c : d$, prove that:

19. $2a + b : b = 2c + d : d$.

20. $3a - b : a = 3c - d : c$.

21. $4a + 3b : 4a - 3b = 4c + 3d : 4c - 3d$.

22. $2a^3 + 3b^3 : 2a^3 - 3b^3 = 2c^3 + 3d^3 : 2c^3 - 3d^3$.

If $a : b = b : c$, prove that:

23. $a^2 + ab : b^2 + bc :: a : c$.

24. $a : c :: (a + b)^2 : (b + c)^2$.

25. If $\dfrac{x - y}{l} = \dfrac{y - z}{m} = \dfrac{z - x}{n}$, and x, y, z are unequal, show that $l + m + n = 0$.

Find x from the proportions:

26. $x+1:x-1=x+2:x-2$.

27. $x+a:2x-b=3x+b:4x-a$.

28. $x^2-4x+2:x^2-2x-1=x^2-4x:x^2-2x-2$.

29. $3+x:4+x=9+x:13+x$.

30. $a+x:b+x=c+x:d+x$.

31. If $a:b=c:d$, show that

$$a^2+b^2:\frac{a^3}{a+b}=c^2+d^2:\frac{c^3}{c+d}.$$

32. When a, b, c, d are proportionals and all unequal, show that no number x can be found such that $a+x$, $b+x$, $c+x$, $d+x$ shall be proportionals.

RATIO OF MAGNITUDES

250. Commensurable Magnitudes. If two magnitudes of the *same kind* are so related that a unit of measure can be found which is contained in each of the magnitudes an integral number of times, this unit of measure is a **common measure** of the two magnitudes, and the two magnitudes are **commensurable.**

Two magnitudes *different in kind* can have no ratio.

If two commensurable magnitudes are measured by the same unit, their **ratio is the ratio of their numerical measures.**

Thus, $\frac{1}{6}$ of a foot is a common measure of $2\frac{1}{2}$ feet and $3\frac{2}{3}$ feet, being contained in the first 15 times and in the second 22 times.

Therefore, the ratio of $2\frac{1}{2}$ feet to $3\frac{2}{3}$ feet is the ratio of $15:22$.

251. Incommensurable Magnitudes. Two magnitudes of the same kind that cannot *both* be expressed in *integers* in terms of a common unit are said to be **incommensurable,** and the *exact value* of their ratio cannot be found. But by taking the unit

sufficiently small, an *approximate value* can be found that shall differ from the true value of the ratio by less than any assigned value, however small.

Suppose a and b to be two incommensurable magnitudes of the *same kind*. Divide b into any integral number, n, of equal parts, and suppose one of these parts is contained in a more than m times and less than $m+1$ times. Then, $\frac{a}{b}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$ and cannot differ from either of these by so much as $\frac{1}{n}$.

But, by increasing n indefinitely, $\frac{1}{n}$ can be made to decrease indefinitely and to become less than any assigned value, however small, though it cannot be made absolutely equal to zero.

Hence, the ratio of two incommensurable magnitudes, although it cannot be expressed *exactly* by numbers, may be expressed *approximately* to any desired degree of accuracy.

Thus, if b represents the length of the side of a square, and a the length of the diagonal, $\frac{a}{b} = \sqrt{2}$.

Now, $\sqrt{2} = 1.41421356 \cdots$, a value greater than 1.414213, but less than 1.414214.

If, then, a *millionth part* of b is taken as the unit, the value of the ratio $\frac{a}{b}$ lies between $\frac{1414213}{1000000}$ and $\frac{1414214}{1000000}$, and therefore differs from either of these fractions by less than $\frac{1}{1000000}$.

By carrying the decimal farther, a fraction may be found that will differ from the true value of the ratio by less than a *billionth*, a *trillionth*, *or by less than any other assigned value whatever.*

Hence, the ratio $\frac{a}{b}$, while it cannot be expressed by numbers *exactly*, may be expressed by numbers *to any degree of accuracy we please.*

252. The ratio of two incommensurable magnitudes is an **incommensurable ratio**, and is *a fixed value* such that an approximate value can be found which will differ from this fixed value by a quantity whose absolute value shall be less than that of any assigned constant, however small.

253. Equal Incommensurable Ratios. As the treatment of Proportion in Algebra depends upon the assumption that it is possible to find fractions which will represent ratios, and as it appears that no fraction can be found to represent exactly the value of an incommensurable ratio, it is necessary to show that *two incommensurable ratios are equal if their approximate values remain equal when the unit of measure is indefinitely diminished.*

Thus, let $a:b$ and $a':b'$ be two incommensurable ratios whose true values lie between the approximate values $\frac{m}{n}$ and $\frac{m+1}{n}$, when the unit of measure is indefinitely diminished. Then they cannot differ from each other by so much as $\frac{1}{n}$.

Let d denote the difference (if any) between $a:b$ and $a':b'$; then

$$d<\frac{1}{n}.$$

Now the true values of $a:b$ and $a':b'$ being fixed, their difference, d, must be fixed, that is, *d must be a constant.*

By increasing n we can make the value of $\frac{1}{n}$ less than any assigned value, however small; hence, $\frac{1}{n}$ can be made less than d if d is not zero.

Therefore, d is 0, and there is no difference between the ratios $a:b$ and $a':b'$. Therefore, $a:b=a':b'$.

254. The laws which apply to the proportion of abstract numbers apply to the proportion of concrete quantities, except that alternation will apply only when the four quantities in proportion are *all* of the same kind.

Exercise 38

1. A rectangular field contains 5270 acres, and its length is to its breadth in the ratio of $31:17$. Find its dimensions.

2. If five gold coins and four silver ones are worth as much as three gold coins and twelve silver ones, find the ratio of the value of a gold coin to that of a silver one.

3. The lengths of two rectangular fields are in the ratio of $2:3$, and the breadths in the ratio of $5:6$. Find the ratio of their areas.

4. Two workmen are paid in proportion to the work they do. A can do in 20 days the work that it takes B 24 days to do. Compare their wages.

5. In a mile race between a bicycle and a tricycle their rates were as $5:4$. The tricycle had half a minute start, but was beaten by 176 yards. Find the rate of each.

6. A railway passenger observes that a train passes him, moving in the opposite direction, in 2 seconds; but moving in the same direction with him, it passes him in 30 seconds. Compare the rates of the two trains.

7. A vessel is half full of a mixture of wine and water. If filled with wine, the ratio of the quantity of wine to that of water is 10 times what it would be if the vessel were filled with water. Find the ratio of the original quantity of wine to that of water.

8. A quantity of milk is increased by watering in the ratio $4:5$, and then 3 gallons are sold; the remainder is increased in the ratio $6:7$ by mixing it with 3 quarts of water. How many gallons of milk were there at first?

9. Each of two vessels, A and B, contains a mixture of wine and water; A in the ratio of $7:3$, and B in the ratio of $3:1$. How many gallons from B must be put with 5 gallons from A to give a mixture of wine and water in the ratio of $11:4$?

10. The time which an express train takes to travel 180 miles is to that taken by an accommodation train as $9:14$. The accommodation train loses as much time from stopping as it would take to travel 30 miles; the express train loses

only half as much time as the other by stopping, and travels 15 miles an hour faster. What are their respective rates?

11. A and B trade with different sums. A gains $200 and B loses $50, and now A's stock is to B's as $2:\frac{1}{2}$. But if A had gained $100 and B lost $85, their stocks would have been as $15:3\frac{1}{2}$. Find the original stock of each.

12. A line is divided into two parts in the ratio $2:3$, and into two parts in the ratio $3:4$; the distance between the points of section is 2. Find the length of the line.

13. A railway consists of two sections; the annual expenditure on one is increased this year 5 per cent, and on the other 4 per cent, producing on the whole an increase of $4\frac{3}{10}$ per cent. Compare the amounts expended on the two sections last year, and also the amounts expended this year.

VARIATION

255. One quantity is said to **vary as** another when the two quantities are so related that the ratio of any two values of the one is equal to the ratio of the corresponding values of the other.

Thus, if it is said that the weight of water varies as its volume, the meaning is that *one* gallon of water is to *any number* of gallons as the weight of *one* gallon is to the weight of the *given number* of gallons.

256. **Function of a Variable.** Two variables may be so related that when a value of one is given the corresponding value of the other can be found. In this case one variable is said to be a *function* of the other.

Thus, if the rate at which a man walks is known, the distance he walks can be found when the time is given; the distance is in this case a *function* of the time.

257. When two variable magnitudes X and Y, not necessarily of the same kind, are so related that when X is changed in

any ratio, Y is changed in the same ratio, Y is said to vary as X, and the relation is denoted thus, $Y \propto X$. The sign $\propto$, called the **sign of variation**, is read *varies as*.

Thus, the area of a triangle with a given base varies as its altitude; for, if the altitude is changed in any ratio, the area is changed in the same ratio.

If $Y \propto X$, and if when X has a definitely assigned value A, Y takes the value B, then

$$B : Y = A : X, \qquad [1]$$

and therefore, by the theory of proportion, B has a value definitely determined by the value of A.

Let the numerical measures of A, B, X, and Y be a, b, x, y, respectively, so that

$$a : x = A : X,$$

and $$b : y = B : Y.$$

Therefore, by [1], $$b : y = a : x.$$

$$\therefore\ b : a = y : x. \qquad [2]$$

Since a and b are the numerical measures of the definitely assigned magnitudes A and B, they are themselves constant and their ratio, $b : a$, is constant. Also, x and y are the numerical measures of the variable magnitudes X and Y; hence, by [2],

When two variable magnitudes X and Y are so related that $Y \propto X$, their numerical measures are so related that their ratio is constant.

Hence, if $y \propto x$, $y : x$ is constant, and if we represent this constant by m,

$$y : x = m : 1, \text{ or } \frac{y}{x} = m. \quad \therefore\ y = mx.$$

Again, if y', x' and y'', x'' are two sets of corresponding values of y and x, then

$$y' : x' = y'' : x'',$$

or $$y' : y'' = x' : x''.$$

258. Inverse Variation. When x and y are so related that the ratio of y to $\frac{1}{x}$ is constant, y is said to vary *inversely* as x; this is written $y \propto \frac{1}{x}$.

Thus, the time required to do a certain amount of work varies inversely as the number of workmen employed; for, if the number of workmen is doubled, halved, or changed in any other ratio, the time required is halved, doubled, or changed in the inverse ratio.

In this case, $$y : \frac{1}{x} = m.$$

$$\therefore\ y = \frac{m}{x}, \text{ and } xy = m;$$

that is, the product xy is constant.

As before, $$y' : \frac{1}{x'} = y'' : \frac{1}{x''},$$

$$x'y' = x''y'',$$

or $$y' : y'' = x'' : x'.$$

259. If the ratio of $y : xz$ is constant, then y is said to *vary jointly* as x and z.

In this case, $$y = mxz,$$

and $$y' : y'' = x'z' : x''z''.$$

260. If the ratio $y : \frac{x}{z}$ is constant, then y varies *directly* as x and *inversely* as z.

In this case, $$y = \frac{mx}{z},$$

and $$y' : y'' = \frac{x'}{z'} : \frac{x''}{z''}.$$

261. Theorem I. If $y \propto x$, and $x \propto z$, then $y \propto z$.

For $$y = mx, \text{ and } x = nz.$$

$$\therefore\ y = mnz.$$

$$\therefore\ y \propto z.$$

262. Theorem II. If $y \propto x$, and $z \propto x$, then $(y \pm z) \propto x$.

For $$y = mx,$$
and $$z = nx.$$
$$\therefore y \pm z = (m \pm n)x.$$
$$\therefore (y \pm z) \propto x.$$

263. Theorem III. If $y \propto x$ when z is constant, and $y \propto z$ when x is constant, and if x and z are independent of each other, then $y \propto xz$ when x and z are both variable.

Let x', y', z' and x'', y'', z'' be two sets of corresponding values of the variables.

Let x change from x' to x'', while z remains constant, and let the corresponding value of y be Y.

Then, $$y' : Y = x' : x''. \qquad [1]$$

Now, let z change from z' to z'', while x remains constant.

Then, $$Y : y'' = z' : z''. \qquad [2]$$

From [1] and [2],
$$y'Y : y''Y = x'z' : x''z'', \qquad (\S\ 248)$$
or $$y' : y'' = x'z' : x''z'',$$
or $$y' : x'z' = y'' : x''z''.$$

$\therefore$ the ratio $\dfrac{y}{xz}$ is constant, and $y \propto xz$.

In like manner, it may be shown that if y varies as each one of any number of independent values $x, z, u, \cdots$, when the rest are unchanged, then when they all change, $y \propto xzu \cdots$

Thus, the area of a rectangle varies as the base when the altitude is constant, and as the altitude when the base is constant, but as the product of the base and altitude when both vary.

The volume of a rectangular solid varies as the length when the width and thickness remain constant; as the width when the length and thickness remain constant; as the thickness when the length and width remain constant; but as the product of length, breadth, and thickness when all three vary.

264. Examples. (1) If y varies inversely as x, and when $y = 2$ the corresponding value of x is 36, find the corresponding value of x when $y = 9$.

Here, $$y = \frac{m}{x}, \text{ or } m = xy.$$
$$\therefore m = 2 \times 36 = 72.$$

If 9 and 72 are substituted for y and m respectively in
$$y = \frac{m}{x},$$
the result is $$9 = \frac{72}{x}, \text{ or } 9x = 72.$$
$$\therefore x = 8.$$

(2) The weight of a sphere of given material varies as its volume, and its volume varies as the cube of its diameter. If a sphere 4 inches in diameter weighs 20 pounds, find the weight of a sphere 5 inches in diameter.

Let W represent the weight,
V represent the volume,
D represent the diameter.

Then, $$W \propto V, \text{ and } V \propto D^3.$$
$$\therefore W \propto D^3.$$
Put $$W = mD^3;$$
then, since 20 and 4 are corresponding values of W and D,
$$20 = m \times 64.$$
$$\therefore m = \tfrac{20}{64} = \tfrac{5}{16}.$$
$$\therefore W = \tfrac{5}{16} D^3.$$
Therefore, when $D = 5$, $W = \frac{5}{16}$ of $125 = 39\frac{1}{16}$.

Exercise 39

1. If $y \propto x$, and $y = 4$ when $x = 5$, find y when $x = 12$.

2. If $y \propto x$, and $y = \frac{1}{3}$ when $x = \frac{1}{2}$, find y when $x = \frac{1}{3}$.

3. If z varies jointly as x and y, and 3, 4, 5 are simultaneous values of x, y, z, find z when $x = y = 10$.

4. If $z \propto \frac{x}{y}$, and $x = 4$ and $y = 3$ when $z = 6$, find the value of z when $x = 5$ and $y = 7$.

5. If the square of x varies inversely as the cube of y, and $x = 2$ when $y = 3$, find the equation between x and y.

6. If z varies as x directly and y inversely, and if $x = 3$ and $y = 4$ when $z = 2$, find z when $x = 15$ and $y = 8$.

7. The velocity acquired by a stone falling from rest varies as the time of falling; and the distance fallen varies as the square of the time. If it is found that in 3 seconds a stone has fallen 145 feet and acquired a velocity of $96\frac{2}{3}$ feet per second, find the velocity and distance fallen at the end of 5 seconds.

8. If a heavier weight draws up a lighter one by means of a string passing over a fixed wheel, the space described in a given time varies directly as the difference between the weights, and inversely as their sum. If 9 ounces draws 7 ounces through 8 feet in 2 seconds, how high will 12 ounces draw 9 ounces in the same time?

9. The space will also vary as the square of the time. Find the space in Example 8, if the time is 3 seconds.

10. Equal volumes of iron and copper are found to weigh 77 and 89 ounces respectively. Find the weight of $10\frac{1}{2}$ feet of round copper rod when 9 inches of iron rod of the same diameter weigh $31\frac{2}{10}$ ounces.

11. The square of the time of a planet's revolution about the sun varies as the cube of its distance from the sun. If the distances of the Earth and Mercury from the sun are as 91 to 35, find in days the time of Mercury's revolution.

12. A spherical iron shell 1 foot in diameter weighs $\frac{91}{216}$ of what it would weigh if solid. Find the thickness of the metal, it being known that the volume of a sphere varies as the cube of its diameter.

CHAPTER XVIII

PROGRESSIONS

265. A succession of numbers that proceed according to some fixed law is called a **series**; the successive numbers are called the **terms** of the series.

A series that ends at some particular term is a **finite series**; a series that continues without end is an **infinite series**.

266. The number of different forms of series is unlimited; in this chapter we shall consider only arithmetical series, geometrical series, and harmonical series.

ARITHMETICAL PROGRESSION

267. A series is called an **arithmetical series** or an **arithmetical progression** when each succeeding term is obtained by adding to the preceding term a *constant difference.*

The general representative of such a series is

$$a, \quad a+d, \quad a+2\,d, \quad a+3\,d, \quad \cdots,$$

in which a is the first term and d the common difference; the series is *increasing* or *decreasing* according as d is positive or negative.

268. The nth Term. Since each succeeding term of the series is obtained by adding d to the preceding term, the coefficient of d is always one less than the number of the term, so that the nth term is $a+(n-1)\,d$.

If the nth term is represented by l, we have

$$\mathbf{l = a + (n-1)\,d.} \qquad \text{(I)}$$

269. Sum of the Series. If l denotes the nth term, a the first term, n the number of terms, d the common difference, and s the sum of n terms, it is evident that

$$\begin{array}{lrl} & s = & a+(a+d)+(a+2d)+\cdots+(l-d)+l \\ \text{or} & s = & l+(l-d)+(l-2d)+\cdots+(a+d)+a \\ \hline & \therefore 2s = & (a+l)+(a+l)+(a+l)+\cdots+(a+l)+(a+l) \\ & = & n(a+l). \end{array}$$

Therefore, $$\mathbf{s = \frac{n}{2}(a + l).} \qquad \text{(II)}$$

270. From the two equations,

$$\mathbf{l = a + (n - 1)d,} \qquad \text{(I)}$$

$$\mathbf{s = \frac{n}{2}(a + l),} \qquad \text{(II)}$$

any *two* of the five numbers a, d, l, n, s may be found when the other *three* are given.

(1) Find the sum of ten terms of the series 2, 5, 8, 11, $\cdots$

Here, $a = 2,\ d = 3,\ n = 10.$

From (I), $l = 2 + 27 = 29.$

Substitute in (II), $s = \frac{10}{2}(2 + 29) = 155.$

(2) The first term of an arithmetical series is 3, the last term 31, and the sum of the series 136. Find the series.

From (I), $31 = 3 + (n - 1)d,$ [1]

From (II), $136 = \frac{n}{2}(3 + 31).$ [2]

From [2], $n = 8.$

Substitute in [1], $d = 4.$

Therefore, the series is 3, 7, 11, 15, 19, 23, 27, 31.

(3) How many terms of the series 5, 9, 13, $\cdots$ must be taken in order that their sum may be 275?

From (I), $l = 5 + (n - 1)4.$

$\therefore l = 4n + 1.$ [1]

From (II), $$275 = \frac{n}{2}(5 + l). \qquad [2]$$

Substitute in [2] the value of l found in [1],

$$275 = \frac{n}{2}(4n + 6),$$

or $$2n^2 + 3n = 275.$$

We now have to solve this quadratic.

Complete the square,

$$16n^2 + (\ ) + 9 = 2209.$$

Extract the root, $$4n + 3 = \pm 47.$$

$$\therefore n = 11 \text{ or } -12\tfrac{1}{2}.$$

We use only the positive result.

(4) Find n when d, l, s are given.

From (I), $$a = l - (n-1)d.$$

From (II), $$a = \frac{2s - ln}{n}.$$

Therefore, $$l - (n-1)d = \frac{2s - ln}{n}.$$

$$\therefore ln - dn^2 + dn = 2s - ln.$$

$$\therefore dn^2 - (2l + d)n = -2s.$$

This is a quadratic with n for the unknown number.

Complete the square,

$$4d^2n^2 - (\ ) + (2l + d)^2 = (2l + d)^2 - 8ds.$$

Extract the root,

$$2dn - (2l + d) = \pm\sqrt{(2l + d)^2 - 8ds}.$$

$$\therefore n = \frac{2l + d \pm \sqrt{(2l + d)^2 - 8ds}}{2d}.$$

NOTE. The table on the next page contains the results of the general solution of all possible problems in arithmetical series, in which three of the numbers a, l, d, n, s are given and two required. The student is advised to work these out, both for the results obtained and for the practice gained in solving literal equations in which the unknown quantities are represented by letters other than x, y, z.

No.	Given	Required	Result
1	$a\ d\ n$	l	$l = a + (n-1)d.$
2	$a\ d\ s$		$l = -\frac{1}{2}d \pm \sqrt{2ds + (a - \frac{1}{2}d)^2}.$
3	$a\ n\ s$		$l = \frac{2s}{n} - a.$
4	$d\ n\ s$		$l = \frac{s}{n} + \frac{(n-1)d}{2}.$
5	$a\ d\ n$	s	$s = \frac{1}{2}n[2a + (n-1)d].$
6	$a\ d\ l$		$s = \frac{l+a}{2} + \frac{l^2 - a^2}{2d}.$
7	$a\ n\ l$		$s = \frac{n}{2}(a + l).$
8	$d\ n\ l$		$s = \frac{1}{2}n[2l - (n-1)d].$
9	$d\ n\ l$	a	$a = l - (n-1)d.$
10	$d\ n\ s$		$a = \frac{s}{n} - \frac{(n-1)d}{2}.$
11	$d\ l\ s$		$a = \frac{1}{2}d \pm \sqrt{(l + \frac{1}{2}d)^2 - 2ds}.$
12	$n\ l\ s$		$a = \frac{2s}{n} - l.$
13	$a\ n\ l$	d	$d = \frac{l-a}{n-1}.$
14	$a\ n\ s$		$d = \frac{2(s - an)}{n(n-1)}.$
15	$a\ l\ s$		$d = \frac{l^2 - a^2}{2s - l - a}.$
16	$n\ l\ s$		$d = \frac{2(nl - s)}{n(n-1)}.$
17	$a\ d\ l$	n	$n = \frac{l-a}{d} + 1.$
18	$a\ d\ s$		$n = \frac{d - 2a \pm \sqrt{(2a-d)^2 + 8ds}}{2d}.$
19	$a\ l\ s$		$n = \frac{2s}{l+a}.$
20	$d\ l\ s$		$n = \frac{2l + d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}.$

271. The **arithmetical mean** between two numbers is the number which, when placed between them, makes with them an arithmetical series.

If a and b represent two numbers, and A their arithmetical mean, then, by the definition of an arithmetical series,

$$A - a = b - A.$$

$$\therefore A = \frac{a+b}{2}.$$

272. Sometimes it is required to insert several arithmetical means between two numbers.

Insert six arithmetical means between 3 and 17.

Here the whole number of terms is 8; 3 is the first term and 17 the eighth.

By (I), $$17 = 3 + 7d.$$

$$\therefore d = 2.$$

Therefore, the complete series is

$$3, \quad [5, \quad 7, \quad 9, \quad 11, \quad 13, \quad 15,] \quad 17,$$

the terms within the brackets being the means required.

273. When the sum of a number of terms in arithmetical progression is given it is convenient to represent the terms as follows:

Three terms by $\quad x - y, \quad x, \quad x + y$;

four terms by $\quad x - 3y, \quad x - y, \quad x + y, \quad x + 3y$;

and so on.

The sum of three numbers in arithmetical progression is 36, and the square of the mean exceeds the product of the two extremes by 49. Find the numbers.

Let $x - y$, x, $x + y$ represent the numbers.

Then, adding, $$3x = 36.$$

$$\therefore x = 12.$$

Putting for x its value, the numbers are $12 - y$, 12, $12 + y$.

By the conditions of the problem,

$$(12)^2 = (12 - y)(12 + y) + 49,$$
$$144 = 144 - y^2 + 49,$$
$$\therefore y = \pm 7.$$

Therefore, the numbers are 5, 12, 19; or 19, 12, 5.

Exercise 40

Find:

1. The tenth term of 3, 8, 13, $\cdots$
2. The eighth term of 12, 9, 6, $\cdots$
3. The twelfth term of -4, -9, -14, $\cdots$
4. The eleventh term of $2\frac{1}{2}$, $1\frac{5}{6}$, $1\frac{1}{6}$, $\cdots$
5. The fourteenth term of $1\frac{1}{3}$, $\frac{1}{4}$, $-\frac{5}{6}$, $\cdots$

Find the sum of:

6. Eight terms of 4, 7, 10, $\cdots$
7. Ten terms of 8, 5, 2, $\cdots$
8. Twelve terms of -3, 1, 5, $\cdots$
9. n terms of 2, $1\frac{1}{6}$, $\frac{1}{3}$, $\cdots$
10. n terms of $2\frac{1}{4}$, $1\frac{5}{6}$, $1\frac{5}{12}$, $\cdots$
11. Given $a = 3$, $l = 55$, $n = 13$. Find d and s.
12. Given $a = 3\frac{1}{4}$, $l = 64$, $n = 82$. Find d and s.
13. Given $a = 1$, $n = 20$, $s = 305$. Find d and l.
14. Given $l = 105$, $n = 16$, $s = 840$. Find a and d.
15. Given $d = 7$, $n = 12$, $s = 594$. Find a and l.
16. Given $a = 9$, $d = 4$, $s = 624$. Find n and l.
17. Given $d = 5$, $l = 77$, $s = 623$. Find a and n.

18. When a train arrives at the top of a long slope the last car is detached and begins to descend, passing over 3 feet in the first second, 3 times 3 feet in the second second, 5 times 3 feet in the third second, and so on. At the end of 2 minutes it reaches the bottom of the slope. What space did the car pass over in the last second?

19. Insert eleven arithmetical means between 1 and 12.

20. The first term of an arithmetical series is 3, and the sum of 6 terms is 28. What term will be 9?

21. How many terms of the series $-5, -2, +1, \cdots$ must be taken in order that their sum may be 63?

22. The arithmetical mean between two numbers is 10, and the mean between the double of the first and the triple of the second is 27. Find the numbers.

23. The first term of an arithmetical progression is 3, the third term is 11. Find the sum of seven terms.

24. Arithmetical means are inserted between 8 and 32, so that the sum of the first two is to the sum of the last two as 7 is to 25. How many means are inserted?

25. In an arithmetical series the common difference is 2, and the square roots of the first, third, and sixth terms form a new arithmetical series. Find the series.

26. Find three numbers in arithmetical progression of which the sum is 21, and the sum of the first and second three-fourths of the sum of the second and third.

27. The sum of three numbers in arithmetical progression is 33, and the sum of their squares is 461. Find the numbers.

28. The sum of four numbers in arithmetical progression is 12, and the sum of their squares 116. What are these numbers?

29. How many terms of the series $1, 4, 7, \cdots$ must be taken in order that the sum of the first half may bear to the sum of the second half the ratio $7 : 22$?

30. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?

31. A man wishes to have his horse shod. The blacksmith asks him $2 a shoe, or 1 cent for the first nail, 3 for the second, 5 for the third, and so on. Each shoe has 8 nails. Ought the man to accept the second proposition?

32. A number consists of three digits which are in arithmetical progression, and this number divided by the sum of its digits is equal to 26; if 198 is added to the number, the digits in the units' and hundreds' places will be interchanged. Required the number.

33. There are placed in a straight line upon a lawn 50 eggs 3 feet distant from each other. A person is required to pick them up one by one and carry them to a basket in the line of the eggs and 3 feet from the first egg, while a runner, starting from the basket, touches a goal and returns. At what distance ought the goal to be placed that both men may have the same distance to pass over?

34. Starting from a box, there are placed upon a straight line 40 stones, at the distances 1 foot, 3 feet, 5 feet, and so on. A man placed at the box is required to take them and carry them back to the box one by one. What is the total distance that he has to accomplish?

35. The sum of five numbers in arithmetical progression is 45, and the product of the first and fifth is five-eighths of the product of the second and fourth. Find the numbers.

GEOMETRICAL PROGRESSION

274. A series is called a **geometrical series** or a **geometrical progression** when each succeeding term is obtained by multiplying the preceding term by a *constant multiplier.*

The general representative of such a series is

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \quad \cdots,$$

in which a is the first term and r the constant multiplier or ratio.

The terms increase or decrease in numerical magnitude according as r is numerically greater than or numerically less than unity.

275. The nth Term. Since the exponent of r increases by one for each succeeding term after the first, the exponent is always one less than the number of the term, so that the nth term is ar^{n-1}.

If the nth term is represented by l, we have

$$\mathbf{l = ar^{n-1}}. \tag{I}$$

276. Sum of the Series. If l represents the nth term, a the first term, n the number of terms, r the common ratio, and s the sum of n terms, then

$$s = a + ar + ar^2 + \cdots + ar^{n-1}. \qquad [1]$$

Multiply by r, $rs = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$. [2]

Subtract the first equation from the second,

$$rs - s = ar^n - a.$$

Resolve each member into its factors,

$$(r-1)s = a(r^n - 1).$$

Divide by $r - 1$.

Therefore, $$\mathbf{s = \frac{a(r^n - 1)}{r - 1}}. \tag{II}$$

Since $l = ar^{n-1}$, $rl = ar^n$, and (II) may be written

$$s = \frac{rl - a}{r - 1}. \qquad \text{(III)}$$

277. From the two formulas (I) and (II), or the two formulas (I) and (III), any *two* of the five numbers a, r, l, n, s may be found when the other *three* are given.

(1) The first term of a geometrical series is 3, the last term 192, and the sum of the series 381. Find the number of terms and the ratio.

From (I), $\quad 192 = 3r^{n-1}. \qquad [1]$

From (III), $\quad 381 = \dfrac{192r - 3}{r - 1}. \qquad [2]$

From [2], $\quad r = 2.$

Substitute in [1], $\quad 2^{n-1} = 64.$

$\therefore n = 7.$

Therefore, the series is 3, 6, 12, 24, 48, 96, 192.

(2) Find l when r, n, s are given.

From (I), $\quad a = \dfrac{l}{r^{n-1}}.$

Substitute in (III), $\quad s = \dfrac{rl - \dfrac{l}{r^{n-1}}}{r - 1}.$

$$(r - 1)s = \frac{(r^n - 1)}{r^{n-1}}l.$$

$$\therefore l = \frac{(r - 1)r^{n-1}s}{r^n - 1}.$$

NOTE. The table on page 212 contains the results of all possible problems in geometrical series in which three of the numbers a, r, l, n, s are given and the other two required, with the exception of those in which n is required; these last require the use of logarithms with which the student is supposed to be not yet acquainted.

No.	Given	Required	Result
1	$a\ r\ n$		$l = ar^{n-1}$.
2	$a\ r\ s$	l	$l = \dfrac{a + (r-1)s}{r}$.
3	$a\ n\ s$		$l(s-l)^{n-1} - a(s-a)^{n-1} = 0$.
4	$r\ n\ s$		$l = \dfrac{(r-1)sr^{n-1}}{r^n - 1}$.
5	$a\ r\ n$		$s = \dfrac{a(r^n - 1)}{r-1}$.
6	$a\ r\ l$		$s = \dfrac{rl - a}{r-1}$.
7	$a\ n\ l$	s	$s = \dfrac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$.
8	$r\ n\ l$		$s = \dfrac{lr^n - l}{r^n - r^{n-1}}$.
9	$r\ n\ l$		$a = \dfrac{l}{r^{n-1}}$.
10	$r\ n\ s$	a	$a = \dfrac{(r-1)s}{r^n - 1}$.
11	$r\ l\ s$		$a = rl - (r-1)s$.
12	$n\ l\ s$		$a(s-a)^{n-1} - l(s-l)^{n-1} = 0$.
13	$a\ n\ l$		$r = \sqrt[n-1]{\dfrac{l}{a}}$.
14	$a\ n\ s$	r	$r^n - \dfrac{s}{a}r + \dfrac{s-a}{a} = 0$.
15	$a\ l\ s$		$r = \dfrac{s-a}{s-l}$.
16	$n\ l\ s$		$r^n - \dfrac{s}{s-l}r^{n-1} + \dfrac{l}{s-l} = 0$.

278. The **geometrical mean** between two numbers is the number which when placed between them makes with them a geometrical series.

If a and b denote two numbers, and G their geometrical mean, then, by the definition of a geometrical series,

$$\frac{G}{a} = \frac{b}{G}.$$

$$\therefore G = \sqrt{ab}.$$

279. Sometimes it is required to insert several geometrical means between two numbers.

Insert three geometrical means between 3 and 48.

Here the whole number of terms is 5; 3 is the first term and 48 the fifth term.

By (I), $48 = 3r^4$.

$\therefore r^4 = 16,$

and $r = \pm 2.$

Therefore, the series is one of the following:

$$3, \; [\; 6, \; 12, \; 24,] \; 48;$$
$$3, \; [-6, \; 12, \; -24,] \; 48.$$

The terms within the brackets are the means required.

280. Infinite Geometrical Series. When r is less than 1, the successive terms become numerically smaller and smaller; by taking n large enough we can make the nth term, ar^{n-1}, as small as we please, although we cannot make it absolutely equal to zero.

The sum of n terms, $\frac{ar^n - a}{r - 1}$, by changing the signs of the numerator and denominator, may be written $\frac{a - ar^n}{1 - r}$, which is equal to $\frac{a}{1-r} - \frac{ar^n}{1-r}$; this sum differs from $\frac{a}{1-r}$ by the fraction $\frac{ar^n}{1-r}$; by taking enough terms we can make ar^n, and consequently this fraction, as small as we please; the greater the number of terms taken the nearer is their sum to $\frac{a}{1-r}$. Hence, $\frac{a}{1-r}$ is called the *sum* of an infinite number of terms of the series.

(1) Find the sum of the infinite series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$

Here $a = 1$, $r = -\frac{1}{2}$.

The sum of the series is $\dfrac{1}{1 + \frac{1}{2}}$, or $\dfrac{2}{3}$.

Therefore, the sum of n terms is

$$\tfrac{2}{3} - \tfrac{2}{3}(-\tfrac{1}{2})^n, \quad \text{or} \quad \tfrac{2}{3}[1 - (-\tfrac{1}{2})^n].$$

This sum evidently approaches $\frac{2}{3}$ as n is increased.

(2) Find the value of the recurring decimal $0.12135135 \cdots$

Consider first the part that recurs; this may be written

$$\tfrac{135}{100000} + \tfrac{135}{100000000} + \cdots,$$

and the sum of this series is $\dfrac{\frac{135}{100000}}{1 - \frac{1}{1000}}$, or $\frac{1}{740}$. Adding 0.12, the part that does not recur, we obtain for the value of the decimal $\frac{449}{3700}$.

Exercise 41

Find:

1. The eighth term of $3, 6, 12, \cdots$
2. The twelfth term of $2, -4, 8, \cdots$
3. The twentieth term of $1, -\frac{1}{3}, \frac{1}{9}, \cdots$
4. The eighteenth term of $3, 2, 1\frac{1}{3}, \cdots$
5. The nth term of $1, -1\frac{1}{3}, 1\frac{7}{9}, \cdots$

Find the sum of:

6. Eleven terms of $4, 8, 16, \cdots$
7. Nineteen terms of $9, 3, 1, \cdots$
8. Twelve terms of $5, -3, 1\frac{4}{5}, \cdots$
9. n terms of $1\frac{7}{8}, \frac{3}{8}, \frac{3}{40}, \cdots$

Sum to infinity:

10. $4 - 2 + 1 - \cdots$
11. $\frac{1}{2} + \frac{1}{3} + \frac{2}{9} + \cdots$
12. $1 - \frac{2}{5} + \frac{4}{25} - \cdots$
13. $\frac{1}{5} + \frac{1}{15} + \frac{1}{45} + \cdots$

Find the value of the recurring decimals:

14. 0.153153 ⋯ **16.** 3.17272 ⋯

15. 0.123535 ⋯ **17.** 4.2561561 ⋯

18. Given $a = 36$, $l = 2\frac{1}{4}$, $n = 5$. Find r and s.

19. Given $l = 128$, $r = 2$, $n = 7$. Find a and s.

20. Given $r = 2$, $n = 7$, $s = 635$. Find a and l.

21. Given $l = 1296$, $r = 6$, $s = 1555$. Find a and n.

22. Insert three geometrical means between 14 and 224.

23. Insert five geometrical means between 2 and 1458.

24. If the first term is 2 and the ratio 3, what term will be 162?

25. The fifth term of a geometrical series is 48, and the ratio 2. Find the first and seventh terms.

26. Four numbers are in geometrical progression; the sum of the first and fourth is 195, and the sum of the second and third is 60. Find the numbers.

27. The sum of four numbers in geometrical progression is 105; the difference between the first and last is to the difference between the second and third in the ratio of 7 : 2. Find the numbers.

28. The first term of an arithmetical progression is 2, and the first, second, and fifth terms are in geometrical progression. Find the sum of 11 terms of the arithmetical progression.

29. The sum of three numbers in arithmetical progression is 6. If 1, 2, 5 are added to the numbers, the three resulting numbers are in geometrical progression. Find the numbers.

30. The sum of three numbers in arithmetical progression is 15; if 1, 4, 19 are added to the numbers, the results are in geometrical progression. Find the numbers.

31. There are four numbers of which the sum is 84; the first three are in geometrical progression and the last three in arithmetical progression; the sum of the second and third is 18. Find the numbers.

32. There are four numbers of which the sum is 13, the fourth being 3 times the second; the first three are in geometrical progression and the last three in arithmetical progression. Find the numbers.

33. The sum of the squares of two numbers exceeds twice their product by 576; the arithmetical mean of the two numbers exceeds the geometrical by 6. Find the numbers.

34. A number consists of three digits in geometrical progression. The sum of the digits is 13; and if 792 is added to the number, the digits in the units' and hundreds' places will be interchanged. Find the number.

35. Find an infinite geometrical series in which each term is 5 times the sum of all the terms that follow it.

36. If a, b, c, d are four numbers in geometrical progression, show that

$$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2.$$

HARMONICAL PROGRESSION

281. A series is called a **harmonical series**, or a **harmonical progression**, when the reciprocals of its terms form an arithmetical series.

The general representative of such a series is

$$\frac{1}{a},\ \frac{1}{a+d},\ \frac{1}{a+2d},\ \ldots,\ \frac{1}{a+(n-1)d}.$$

Questions relating to harmonical series are best solved by writing the reciprocals of its terms, and thus forming an arithmetical series.

282. The **harmonical mean** between two numbers is the number which when placed between them makes with them a harmonical series.

If a and b denote two numbers, and H their harmonical mean, then, by the definition of a harmonical series,

$$\frac{1}{H}-\frac{1}{a}=\frac{1}{b}-\frac{1}{H}.$$

$$\therefore \frac{2}{H}=\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}.$$

$$\therefore H=\frac{2ab}{a+b}.$$

283. Sometimes it is required to insert several harmonical means between two numbers.

Insert three harmonical means between 3 and 18.

Find the three arithmetical means between $\frac{1}{3}$ and $\frac{1}{18}$.

These are found to be $\frac{19}{72}$, $\frac{14}{72}$, $\frac{9}{72}$; therefore, the harmonical means are $\frac{72}{19}$, $\frac{72}{14}$, $\frac{72}{9}$; or $3\frac{15}{19}$, $5\frac{1}{7}$, 8.

284. Since $A=\dfrac{a+b}{2}$, and $G=\sqrt{ab}$,

$$H=\frac{G^2}{A}, \text{ or } G=\sqrt{AH}.$$

That is, the geometrical mean between two numbers is also the geometrical mean between the arithmetical and harmonical means of the numbers, or

$$A:G=G:H.$$

Hence, G lies in numerical value between A and H.

Exercise 42

1. Insert four harmonical means between 2 and 12.

2. Find two numbers whose difference is 8 and the harmonical mean between them $1\frac{4}{5}$.

3. Find the seventh term of the harmonical series 3, $3\frac{3}{7}$, 4, ...

4. Continue to two terms each way the harmonical series of which two consecutive terms are 15, 16.

5. The first two terms of a harmonical series are 5 and 6. What term will be 30?

6. The fifth and ninth terms of a harmonical series are 8 and 12. Find the first four terms.

7. The difference between the arithmetical and harmonical means between two numbers is $1\frac{4}{5}$, and one of the numbers is 4 times the other. Find the numbers.

8. The arithmetical mean between two numbers exceeds the geometrical by 13, and the geometrical exceeds the harmonical by 12. What are the numbers?

9. The sum of three terms of a harmonical series is 39, and the third is the product of the other two. Find the terms.

10. When a, b, c are in harmonical progression show that $a:c = a-b:b-c$.

11. If a and b are positive, which is the greater, A or H?

12. Show that a, b, and c will be in arithmetical progression, in geometrical progression, or in harmonical progression, according as $a-b:b-c$ is equal to $a:a$, to $a:b$, or to $a:c$.

CHAPTER XIX

BINOMIAL THEOREM; POSITIVE INTEGRAL EXPONENT

285. Binomial Theorem; Positive Integral Exponent. By successive multiplications we obtain the following identities:

$$(a+b)^2 \equiv a^2 + 2\,ab + b^2;$$
$$(a+b)^3 \equiv a^3 + 3\,a^2b + 3\,ab^2 + b^3;$$
$$(a+b)^4 \equiv a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4.$$

The expressions on the right may be written in a form better adapted to show the law of their formation:

$$(a+b)^2 \equiv a^2 + 2\,ab + \frac{2\cdot 1}{1\cdot 2}b^2;$$
$$(a+b)^3 \equiv a^3 + 3\,a^2b + \frac{3\cdot 2}{1\cdot 2}ab^2 + \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}b^3;$$
$$(a+b)^4 \equiv a^4 + 4\,a^3b + \frac{4\cdot 3}{1\cdot 2}a^2b^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}ab^3 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}b^4.$$

286. Let n represent the exponent of $(a+b)$ in any one of these identities; then, in the expressions on the right, we observe that the following laws hold true:

1. The number of terms is $n+1$.

2. The first term is a^n, and the exponent of a decreases by one in each succeeding term. The first power of b occurs in the second term, the second power in the third term, and the exponent of b increases by one in each succeeding term.

The sum of the exponents of a and b in any term is n.

3. The coefficient of the first term is 1; of the second term, n; of the third term, $\frac{n(n-1)}{1\cdot 2}$; and so on.

287. The Coefficient of Any Term. The number of factors in the numerator of the coefficient of any term is the same as the number of factors in the denominator of that term. The number of factors in each numerator and denominator is the same as the exponent of b in that term, and this exponent is one less than the number of the term.

288. Proof of the Theorem. Show that the laws of § 286 hold true when the exponent is *any* positive integer.

We know that the laws hold for the fourth power; suppose, for the moment, that they hold for the kth power, k being any positive integer.

We shall then have

$$(a+b)^k \equiv a^k + ka^{k-1}b + \frac{k(k-1)}{1\cdot 2}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}a^{k-3}b^3 + \cdots \qquad [1]$$

Multiply both members of [1] by $a+b$; the result is

$$(a+b)^{k+1} \equiv a^{k+1} + (k+1)a^kb + \frac{(k+1)k}{1\cdot 2}a^{k-1}b^2 + \frac{(k+1)k(k-1)}{1\cdot 2\cdot 3}a^{k-2}b^3 + \cdots \qquad [2]$$

In the right member of [1] for k put $k+1$; this gives

$$a^{k+1} + (k+1)a^kb + \frac{(k+1)(k+1-1)}{1\cdot 2}a^{k-1}b^2 + \frac{(k+1)(k+1-1)(k+1-2)}{1\cdot 2\cdot 3}a^{k-2}b^3 + \cdots$$

This last expression, simplified, is seen to be identical with the right member of [2], and this in turn by [2] is identical with $(a+b)^{k+1}$.

Hence, [1] holds when for k we put $k+1$; that is, if the laws of § 286 hold for the kth power, they must hold for the $(k+1)$th power.

But the laws hold for the fourth power; therefore, they must hold for the fifth power.

Holding for the fifth power, they must hold for the sixth power; and so on for any positive integral power.

Therefore, they must hold for the nth power if n is a positive integer; and we have

$$(a+b)^n \equiv a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots \qquad [A]$$

289. This formula is known as the **binomial theorem.**

The expression on the right is known as the **expansion** of $(a+b)^n$; this expansion is a *finite series* when n is a positive integer. That the series is finite may be seen as follows:

In writing the successive coefficients we shall finally arrive at a coefficient which contains the factor $n-n$; the corresponding term will vanish. The coefficients of the succeeding terms likewise all contain the factor $n-n$, and all these terms will vanish.

290. If a and b are interchanged, the identity [A] is written

$$(a+b)^n \equiv (b+a)^n \equiv b^n + nb^{n-1}a + \frac{n(n-1)}{1\cdot 2}b^{n-2}a^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}b^{n-3}a^3 + \cdots$$

This last expansion is the expansion of [A] written in reverse order. Comparing the two expansions, we see that: the coefficient of the last term is the same as the coefficient of the first term; the coefficient of the last term but one is the same as the coefficient of the first term but one; and so on.

In general, the coefficient of the rth term from the end is the same as the coefficient of the rth term from the beginning.

In writing an expansion by the binomial theorem, after arriving at the middle term, we can shorten the work by observing that the remaining coefficients are those already found, written in reverse order.

291. If b is negative, the terms which involve *even* powers of b are positive; and the terms which involve *odd* powers of b are negative. Hence,

$$(\mathbf{a}-\mathbf{b})^{\mathbf{n}} \equiv \mathbf{a}^{\mathbf{n}} - \mathbf{n}\mathbf{a}^{\mathbf{n}-1}\mathbf{b} + \frac{\mathbf{n}(\mathbf{n}-1)}{1\cdot 2}\mathbf{a}^{\mathbf{n}-2}\mathbf{b}^2 - \frac{\mathbf{n}(\mathbf{n}-1)(\mathbf{n}-2)}{1\cdot 2\cdot 3}\mathbf{a}^{\mathbf{n}-3}\mathbf{b}^3 + \cdots \qquad [B]$$

If we put 1 for a and x for b in [A],

$$(1+x)^n \equiv 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots \qquad [C]$$

If we put 1 for a and x for b in [B],

$$(1-x)^n \equiv 1 - nx + \frac{n(n-1)}{1\cdot 2}x^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots \qquad [D]$$

292. Examples. (1) Expand $(1+2x)^5$.

In [C] put $2x$ for x and 5 for n. The result is

$$\begin{aligned}(1+2x)^5 &\equiv 1 + 5(2x) + \frac{5\cdot 4}{1\cdot 2}(2x)^2 + \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}(2x)^3 \\ &\quad + \frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}(2x)^4 + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5}(2x)^5 \\ &\equiv 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5.\end{aligned}$$

(2) Expand to three terms $\left(\frac{1}{x}-\frac{2x^2}{3}\right)^6$.

Put a for $\frac{1}{x}$, and b for $\frac{2x^2}{3}$.

Then, by [B], $(a-b)^6 \equiv a^6 - 6a^5b + 15a^4b^2 - \cdots$

Replacing a and b by their values,

$$\left(\frac{1}{x}-\frac{2x^2}{3}\right)^6 \equiv \left(\frac{1}{x}\right)^6 - 6\left(\frac{1}{x}\right)^5\left(\frac{2x^2}{3}\right) + 15\left(\frac{1}{x}\right)^4\left(\frac{2x^2}{3}\right)^2 - \cdots$$

$$\equiv \frac{1}{x^6} - \frac{4}{x^3} + \frac{20}{3} - \cdots$$

293. Any Required Term. From [A] it is evident (§ 286) that the $(r+1)$th term in the expansion of $(a+b)^n$ is

$$\frac{n(n-1)(n-2)\cdots \text{to } r \text{ factors}}{1\cdot 2\cdot 3\cdots r}a^{n-r}b^r.$$

The $(r+1)$th term in the expansion of $(a-b)^n$ is the same as the above if r is even, and the negative of the above if r is odd.

Find the eighth term of $\left(4-\frac{x^2}{2}\right)^{10}$.

Here, $a=4$, $b=\frac{x^2}{2}$, $n=10$, $r=7$.

The eighth term is $-\frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(4)^3\left(\frac{x^2}{2}\right)^7$, or $-60x^{14}$.

294. The Greatest Coefficient. Suppose that the coefficient of the $(r+1)$th term is the numerically greatest coefficient.

This coefficient and the preceding and following coefficients are as follows:

rth term, $$\frac{n(n-1)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)};$$

$(r+1)$th term, $$\frac{n(n-1)\cdots(n-r+2)(n-r+1)}{1\cdot 2\cdot 3\cdots(r-1)r};$$

$(r+2)$th term, $$\frac{n(n-1)\cdots(n-r+2)(n-r+1)(n-r)}{1\cdot 2\cdot 3\cdots(r-1)r(r+1)}.$$

The coefficient of the rth term may be obtained by multiplying the coefficient of the $(r+1)$th by $\frac{r}{n-r+1}$; the coefficient of the $(r+2)$th, by multiplying the coefficient of the $(r+1)$th by $\frac{n-r}{r+1}$. If the coefficient of the $(r+1)$th is numerically the greatest,

$$\frac{r}{n-r+1}<1, \qquad \text{and } \frac{n-r}{r+1}<1.$$

Therefore, $\quad r<n-r+1$, and $r+1>n-r$.

Therefore, $\quad r<\frac{n+1}{2}, \qquad$ and $\qquad r>\frac{n-1}{2}$.

If n is even, $r=\frac{n}{2}$, and $r+1=\frac{n+2}{2}$; in this case there is one middle term, and its coefficient is the greatest coefficient.

If n is odd, we can have only $r=\frac{n+1}{2}$, or $r=\frac{n-1}{2}$; in in this case there are two middle terms; their coefficients are alike and are the two greatest coefficients.

295. A trinomial may be expanded by the binomial theorem as follows:

Expand $(1+2x-x^2)^3$.

$$\begin{aligned}\therefore (1+2x-x^2)^3 &\equiv 1+3(2x-x^2)+3(2x-x^2)^2+(2x-x^2)^3\\ &\equiv 1+6x+9x^2-4x^3-9x^4+6x^5-x^6.\end{aligned}$$

Exercise 43

Expand:

1. $(1+3x)^5$.
2. $\left(1+\frac{2x}{3}\right)^4$.
3. $\left(1-\frac{\sqrt{x^3}}{3}\right)^4$.
4. $(2+x^2)^6$.
5. $\left(\frac{2}{x}-\frac{x^2}{4}\right)^5$.
6. $\left(\frac{2a}{x}-\frac{x^2}{(2a)^2}\right)^5$.
7. $(3x-2y)^6$.
8. $\left(\frac{2x^2}{y}-\frac{\sqrt[3]{y^2}}{4}\right)^5$.
9. $\left(\sqrt{\frac{a^3}{b^2}}-\frac{\sqrt[4]{b^3}}{4a}\right)^4$.
10. $(1+4x+3x^2)^4$.
11. $(a^2-ax-2x^2)^3$.

Find:

12. The fourth term of $\left(x + \dfrac{1}{2x}\right)^8$.

13. The eighth term of $\left(2 - \dfrac{1}{4x^2}\right)^{10}$.

14. The twelfth term of $\left(\dfrac{1}{x} - \dfrac{\sqrt{x}}{4}\right)^{14}$.

15. The twentieth term of $\left(x - \dfrac{2}{3\sqrt[4]{x}}\right)^{23}$.

16. The fourteenth term of $\left(\sqrt[3]{x^2} - \dfrac{1}{2\sqrt{x}}\right)^{17}$.

17. The $(r+1)$th term of $\left(\sqrt{x} + \sqrt[3]{\dfrac{3}{2x}}\right)^8$.

18. The $(r+1)$th term of $\left(\sqrt{\dfrac{1}{3x}} - \dfrac{\sqrt{x}}{2}\right)^{10}$.

19. The $(r+3)$th term of $\left(\dfrac{x}{2y} - \dfrac{y}{\sqrt{3x}}\right)^{12}$.

20. The middle term of $\left(\dfrac{3}{4x} - \sqrt{\dfrac{x^3}{2}}\right)^{12}$.

21. The two middle terms of $\left(\dfrac{a}{\sqrt{2x}} + \sqrt{\dfrac{3x}{4a}}\right)^{15}$.

22. The rth term from the end of $\left(\dfrac{\sqrt[3]{x^2}}{4} - \sqrt{\dfrac{x^3}{2}}\right)^{11}$.

23. In the expansion of $(a+b)^n$ show that the sum of the coefficients is 2^n.

24. In the expansion of $(a-b)^n$ show that the sum of the positive coefficients equals the sum of the negative coefficients.

25. Expand

$$\left(x + \frac{\sqrt{-1}}{2x}\right)^4;\ \left(\sqrt{-1} + \frac{\sqrt[3]{x}}{4\sqrt{-1}}\right)^5;\ \left(\frac{\sqrt{-a}}{2} + \frac{1}{a\sqrt{-1}}\right)^5.$$

CHAPTER XX

LOGARITHMS

296. Definitions. Let any positive number except 1 be selected as a **base**. Then, the index or exponent which the base must have to produce a given number is called the **logarithm** of that number to the given base.

Any positive number except 1 may be selected as the base; and to each base corresponds a **system of logarithms**.

Thus, since $2^3 = 8$, the logarithm of 8 in the system of which 2 is the base is 3.

That is, the logarithm of 8 to the base 2 is 3; this is abbreviated $\log_2 8 = 3$.

In general, if $a^n = N$, then $n = \log_a N$.

Observe that $a^n = N$ and $n = \log_a N$ are two different ways of expressing the same relation between n and N. The identity, $a^{\log_a N} \equiv N$, is sometimes useful.

The subscript which shows the base is usually omitted when there is no uncertainty as to what number is being used as the base.

In this chapter only the positive scalar values of the root will be considered; consequently, in a system with a positive base, negative numbers cannot have scalar logarithms.

297. The logarithms of such numbers as are perfect powers of the base selected are commensurable numbers; the logarithms of all other numbers are incommensurable numbers.

REMARK. By an incommensurable number is meant a number that has no common measure with unity (§ 251).

Incommensurable logarithms are expressed approximately to any desired degree of accuracy by means of decimal fractions.

298. A logarithm in general consists of two parts, an integral part and a fractional part; the integral part is called the **characteristic**, and the fractional part the **mantissa**.

The calculation of logarithms to a given base will be considered in Chapter XXV.

299. Incommensurable Exponents. It will now be necessary to prove that the laws which in Chapter IX were found to apply to commensurable exponents apply also to incommensurable exponents.

Let a be *any positive number except* 1, and let m and n be two positive incommensurable numbers.

To prove $$a^m a^n = a^{m+n}.$$

We can always find (§ 251) four positive integers, p, q, r, s, such that m lies between $\frac{p}{q}$ and $\frac{p+1}{q}$, and n between $\frac{r}{s}$ and $\frac{r+1}{s}$.

Then, a^m lies between $a^{\frac{p}{q}}$ and $a^{\frac{p+1}{q}}$, and a^n lies between $a^{\frac{r}{s}}$ and $a^{\frac{r+1}{s}}$.

Therefore, $a^m a^n$ lies between $a^{\frac{p}{q}}a^{\frac{r}{s}}$ and $a^{\frac{p+1}{q}}a^{\frac{r+1}{s}}$.

But $$a^{\frac{p}{q}}a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}},$$

and $$a^{\frac{p+1}{q}}a^{\frac{r+1}{s}} = a^{\frac{p}{q}+\frac{r}{s}+\frac{1}{q}+\frac{1}{s}}.$$

Hence, $a^m a^n$ lies between $a^{\frac{p}{q}+\frac{r}{s}}$ and $a^{\frac{p}{q}+\frac{r}{s}+\frac{1}{q}+\frac{1}{s}}$, and consequently differs from $a^{\frac{p}{q}+\frac{r}{s}}$ by less than $\left(a^{\frac{p}{q}+\frac{r}{s}+\frac{1}{q}+\frac{1}{s}} - a^{\frac{p}{q}+\frac{r}{s}}\right)$; that is, by less than $a^{\frac{p}{q}+\frac{r}{s}}\left(a^{\frac{1}{q}+\frac{1}{s}} - 1\right)$.

Also, since m lies between $\frac{p}{q}$ and $\frac{p+1}{q}$, and n between $\frac{r}{s}$ and $\frac{r+1}{s}$, a^{m+n} lies between $a^{\frac{p}{q}+\frac{r}{s}}$ and $a^{\frac{p}{q}+\frac{r}{s}+\frac{1}{q}+\frac{1}{s}}$, and consequently differs from $a^{\frac{p}{q}+\frac{r}{s}}$ by less than $a^{\frac{p}{q}+\frac{r}{s}}\left(a^{\frac{1}{q}+\frac{1}{s}} - 1\right)$.

Therefore, the expressions $a^m a^n$ and a^{m+n} have the same approximate value $a^{\frac{p}{q}+\frac{r}{s}}$, and each differs from this value by less than $a^{\frac{p}{q}+\frac{r}{s}}(a^{\frac{1}{q}+\frac{1}{s}}-1)$.

Now let q and s be continually increased, p and r being always so taken that m lies between $\frac{p}{q}$ and $\frac{p+1}{q}$, and n between $\frac{r}{s}$ and $\frac{r+1}{s}$. Then, $\frac{1}{q}$ and $\frac{1}{s}$ continually decrease; $a^{\frac{1}{q}+\frac{1}{s}}$ approximates to a^0 or 1; and $a^{\frac{p}{q}+\frac{r}{s}}(a^{\frac{1}{q}+\frac{1}{s}}-1)$ continually decreases.

Therefore, the difference between $a^m a^n$ and $a^{\frac{p}{q}+\frac{r}{s}}$ continually decreases; the difference between a^{m+n} and $a^{\frac{p}{q}+\frac{r}{s}}$ continually decreases; and each difference becomes as small as we please.

But, however great q and s may be, the expressions $a^m a^n$ and a^{m+n} have the same approximate value, $a^{\frac{p}{q}+\frac{r}{s}}$.

Therefore, as in § 253, we must have

$$a^m a^n = a^{m+n}.$$

The foregoing proof is easily extended to the case in which m and n are one or both negative.

Having proved for incommensurable exponents that

$$a^m a^n = a^{m+n},$$

it is easily proved that

$$\frac{a^m}{a^n} = a^{m-n};\ (a^m)^n = a^{mn};\ \sqrt[n]{a^m} = a^{\frac{m}{n}};\ a^m b^m = (ab)^m.$$

300. Properties of Logarithms. Let a be the base, M and N any positive numbers, m and n their logarithms to the base a; so that

$$a^m = M, \qquad a^n = N,$$
$$m = \log_a M, \qquad n = \log_a N.$$

Then, in any system of logarithms:

1. *The logarithm of* 1 *is* 0.

For, $\quad a^0 = 1. \qquad \therefore 0 = \log_a 1.$

2. *The logarithm of the base itself is* 1.

For, $$a^1 = a. \qquad \therefore 1 = \log_a a.$$

3. *The logarithm of the reciprocal of a positive number is the negative of the logarithm of the number.*

For, if $$a^n = N, \text{ then } \frac{1}{N} = \frac{1}{a^n} = a^{-n}.$$

$$\therefore \log_a \frac{1}{N} = -n = -\log_a N.$$

4. *The logarithm of the product of two or more positive numbers is the sum of the logarithms of the several factors.*

For, $$M \times N = a^m \times a^n = a^{m+n}.$$

$$\therefore \log_a (M \times N) = m + n = \log_a M + \log_a N.$$

Similarly for the product of three or more factors.

5. *The logarithm of the quotient of two positive numbers is the remainder found by subtracting the logarithm of the divisor from the logarithm of the dividend.*

For, $$\frac{M}{N} = \frac{a^m}{a^n} = a^{m-n}.$$

$$\therefore \log_a \frac{M}{N} = m - n = \log_a M - \log_a N.$$

6. *The logarithm of a power of a positive number is the product of the logarithm of the number by the exponent of the power.*

For, $$N^p = (a^n)^p = a^{np}.$$

$$\therefore \log_a N^p = np = p \log_a N.$$

7. *The logarithm of the real positive value of a root of a positive number is the quotient found by dividing the logarithm of the number by the index of the root.*

For, $$\sqrt[r]{N} = \sqrt[r]{a^n} = a^{\frac{n}{r}}.$$

$$\therefore \log_a \sqrt[r]{N} = \frac{n}{r} = \frac{\log_a N}{r}.$$

301. In a system with a base greater than 1 the logarithms of all positive numbers greater than 1 are positive, and the logarithms of all positive numbers less than 1 are negative.

Conversely, in a system with a positive base less than 1 the logarithms of all positive numbers greater than 1 are negative, and the logarithms of all positive numbers less than 1 are positive.

302. Two Important Systems. Although the possible number of different systems of logarithms is unlimited, there are but two systems in common use. These are:

1. The **common system**, also called the **Briggs, denary,** or **decimal** system, of which the base is 10.

2. The **natural system**, of which the base is the *natural base.*

The natural base, generally represented by e, is the fixed value which the sum of the series

$$1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\cdots$$

approaches as the number of terms is indefinitely increased. The value of e, carried to seven places of decimals, is

$$2.7182818\cdots$$

The common system is the system used in actual calculation; the natural system is used in the higher mathematics.

303. Common Logarithms. By logarithms in §§ 303–317 we mean the common logarithms.

Since
$$\begin{aligned} 10^0 &= 1, & 10^{-1}\left(=\tfrac{1}{10}\right) &= 0.1,\\ 10^1 &= 10, & 10^{-2}\left(=\tfrac{1}{100}\right) &= 0.01,\\ 10^2 &= 100, & 10^{-3}\left(=\tfrac{1}{1000}\right) &= 0.001,\end{aligned}$$

therefore
$$\begin{aligned} \log 1 &= 0, & \log 0.1 &= -1,\\ \log 10 &= 1, & \log 0.01 &= -2,\\ \log 100 &= 2, & \log 0.001 &= -3.\end{aligned}$$

Also, it is evident that the common logarithm of any number between

1	and	10	will be	$0 +$ a fraction,
10	and	100	will be	$1 +$ a fraction,
100	and	1000	will be	$2 +$ a fraction,
1	and	0.1	will be	$-1 +$ a fraction,
0.1	and	0.01	will be	$-2 +$ a fraction,
0.01	and	0.001	will be	$-3 +$ a fraction.

304. With common logarithms the mantissa is always made *positive.* Hence, in the case of numbers less than 1 whose logarithms are *negative,* the logarithm is made to consist of a *negative* characteristic and a *positive* mantissa.

When a logarithm consists of a *negative* characteristic and a *positive* mantissa it is usual to write the minus sign *over* the characteristic, or to add 10 to the characteristic and to indicate the subtraction of 10 from the resulting logarithm.

Thus, $\log 0.2 = \bar{1}.3010$, and this may be written $9.3010 - 10$.

305. The *characteristic* of the logarithm of an integral number, or of a mixed number, is *one less* than the number of integral digits in the number.

Thus, from § 303, $\log 1 = 0$, $\log 10 = 1$, $\log 100 = 2$. Hence, the common logarithms of all numbers from 1 to 10 (that is, of all numbers consisting of *one* integral digit) have 0 for characteristic; and the common logarithms of all numbers from 10 to 100 (that is, of all numbers consisting of *two* integral digits) have 1 for characteristic; and so on, the characteristic increasing by one for each increase of one in the number of digits, and hence being always *one less than the number of integral digits.*

306. The *characteristic* of the common logarithm of a decimal fraction is *negative,* and is equal to the number of the place occupied by the first significant figure of the decimal.

Thus, from § 303, $\log 0.1 = -1$, $\log 0.01 = -2$, $\log 0.001 = -3$. Hence, the common logarithms of all numbers from 0.1 to 1 have -1

for characteristic (the mantissa being *positive*), the common logarithms of all numbers from 0.01 to 0.1 have -2 for characteristic, the common logarithms of all numbers from 0.001 to 0.01 have -3 for characteristic, and so on; the characteristic always being *negative and equal to the number of the place occupied by the first significant figure of the decimal.*

307. The *mantissa* of the common logarithm of any integral number, or decimal fraction, depends only upon the sequence of the digits of the number, and is unchanged so long as the *sequence of the digits* remains the same.

For, changing the position of the decimal point in a number is equivalent to multiplying or dividing the number by a power of 10. Its common logarithm, therefore, is increased or diminished by the *exponent* of that power of 10; and, since this exponent is *integral*, the *mantissa*, or decimal part of the logarithm, is unaffected.

Thus,
$$27{,}196 = 10^{4.4345}, \qquad 2.7196 = 10^{0.4345},$$
$$2719.6 = 10^{3.4345}, \qquad 0.27196 = 10^{9.4345-10},$$
$$27.196 = 10^{1.4345}, \qquad 0.0027196 = 10^{7.4345-10}.$$

One advantage of using the number *ten* as the base of a system of logarithms consists in the fact that the *mantissa* depends only on the *sequence of digits*, and the *characteristic* depends only on the *position of the decimal point.*

308. In simplifying the logarithm of a root the equal positive and negative numbers to be added to the logarithm should be such that the resulting negative number, when divided by the index of the root, gives a quotient of -10.

Thus, $$\log 0.002^{\frac{1}{3}} = \tfrac{1}{3} \text{ of } (7.3010 - 10).$$

The expression $$\tfrac{1}{3} \text{ of } (7.3010 - 10)$$

may be put in the form $\frac{1}{3}$ of $(27.3010 - 30)$, which is $9.1003 - 10$, since the addition of 20 to the 7, and of -20 to the -10, produces no change in the *value* of the logarithm.

Again,
$$\log 0.0002^{\frac{1}{5}} = \tfrac{1}{5} \text{ of } (6.3010 - 10)$$
$$= \tfrac{1}{5} \text{ of } (46.3010 - 50)$$
$$= 9.2602 - 10.$$

Exercise 44

Given: $\log 2 = 0.3010$; $\log 3 = 0.4771$; $\log 5 = 0.6990$; $\log 7 = 0.8451$.

Find the common logarithms of the following numbers by resolving the numbers into factors and taking the sum of the logarithms of the factors:

1. 6.	5. 25.	9. 0.021.	13. 2.1.
2. 15.	6. 30.	10. 0.35.	14. 16.
3. 21.	7. 42.	11. 0.0035.	15. 0.056.
4. 14.	8. 420.	12. 0.004.	16. 0.63.

Find the common logarithm of:

17. 2^3.
18. 5^2.
19. 7^4.
20. 5^5.
21. $2^{\frac{1}{2}}$.
22. $5^{\frac{1}{2}}$.
23. $5^{\frac{1}{5}}$.
24. $7^{\frac{1}{10}}$.
25. $2^{\frac{3}{4}}$.
26. $7^{\frac{2}{7}}$.
27. $5^{\frac{3}{5}}$.
28. $3^{\frac{2}{11}}$.
29. $5^{\frac{3}{4}}$.
30. $2^{\frac{11}{4}}$.
31. $5^{\frac{2}{3}}$.
32. $\frac{2}{5}$.
33. $\frac{2}{7}$.
34. $\frac{3}{5}$.
35. $\frac{3}{7}$.
36. $\frac{5}{3}$.
37. $\frac{5}{2}$.
38. $\frac{7}{3}$.
39. $\frac{7}{2}$.
40. $\dfrac{0.05}{3}$.
41. $\dfrac{0.005}{2}$.
42. $\dfrac{0.07}{5}$.
43. $\dfrac{5}{0.07}$.
44. $\dfrac{0.05}{0.003}$.
45. $\dfrac{0.007}{0.02}$.
46. $\dfrac{0.02}{0.007}$.
47. $\dfrac{0.005}{0.07}$.
48. $\dfrac{0.02^2}{3^3}$.
49. $\dfrac{3^3}{0.02^2}$.
50. $\dfrac{7^3}{0.02^2}$.
51. $\dfrac{0.07^3}{0.003^3}$.

309. The remainder obtained by subtracting the logarithm of a number from 10 is called the **cologarithm** of the number, or **arithmetical complement** of the logarithm of the number.

The cologarithm is abbreviated colog, and is most easily found by beginning with the characteristic of the logarithm

and subtracting each figure from 9 down to the last significant figure, and subtracting that figure from 10.

Thus, $\log 7 = 0.8451$; and $\text{colog } 7 = 9.1549$. We readily find colog 7 by subtracting, mentally, 0 from 9, 8 from 9, 4 from 9, 5 from 9, 1 from 10, and writing the resulting figure at each step.

310. If 10 is subtracted from the cologarithm of a number, the result is the logarithm of the reciprocal of that number.

Thus,
$$\begin{aligned}\log \frac{1}{N} &= \log 1 - \log N\\ &= 0 - \log N\\ &= (10 - \log N) - 10\\ &= \text{colog } N - 10.\end{aligned}$$

311. The addition of a (cologarithm $-$ 10) is equivalent to the subtraction of a logarithm.

Thus,
$$\text{colog } N - 10 = (10 - \log N) - 10 = -\log N.$$

312. The logarithm of a quotient may be found by *adding* the *logarithm* of the dividend and the *cologarithm* of the divisor, and subtracting 10 from the result.

In finding a cologarithm when the *characteristic* of the logarithm is a *negative* number, it must be observed that the *subtraction* of a *negative* number is equivalent to the *addition* of an *equal positive* number.

Thus,
$$\begin{aligned}\log \frac{5}{0.002} &= \log 5 + \text{colog } 0.002 - 10\\ &= 0.6990 + 12.6990 - 10\\ &= 3.3980.\end{aligned}$$

Here, $\log 0.002 = \bar{3}.3010$, and in subtracting -3 from 9 the result is the same as adding $+3$ to 9.

Again,
$$\begin{aligned}\log \frac{2}{0.07} &= \log 2 + \text{colog } 0.07 - 10\\ &= 0.3010 + 11.1549 - 10\\ &= 1.4559.\end{aligned}$$

Also,
$$\begin{aligned}\log \frac{0.07}{2^3} &= 8.8451 - 10 + 9.0970 - 10\\ &= 17.9421 - 20\\ &= 7.9421 - 10.\end{aligned}$$

Here, $\log 2^3 = 3 \log 2 = 3 \times 0.3010 = 0.9030.$

Hence, $\text{colog } 2^3 = 10 - 0.9030 = 9.0970.$

313. Tables. A table of *four-place* common logarithms is given at the end of this chapter, which contains the common logarithms of all numbers under 1000, *the decimal point and characteristic being omitted.* The logarithms of single digits 1, 8, etc., are found at 10, 80, etc.

Tables that contain logarithms of more places can be procured, but this table will serve for many practical uses, and will enable the student to use tables of five-place, seven-place, and ten-place logarithms in work that requires greater accuracy.

In working with a four-place table, the numbers corresponding to the logarithms, that is, the *antilogarithms,* as they are called, may be carried to *four significant digits.*

314. To find the Logarithm of a Number in this Table.

(1) Find the logarithm of 65.7.

In the column headed "N" look for the first two significant figures, and at the top of the table for the third significant figure. In the line with 65, and in the column headed 7, is seen 8176. To this number prefix the characteristic and insert the decimal point. Thus,

$$\log 65.7 = 1.8176.$$

(2) Find the logarithm of 20,347.

In the line with 20, and in the column headed 3, is seen 3075; also in the line with 20, and in the 4 column, is seen 3096, and the difference between these two is 21. The difference between 20,300 and 20,400 is 100, and the difference between 20,300 and 20,347 is 47. Hence, $\frac{47}{100}$ of $21 = 10$, nearly, must be added to 3075. That is,

$$\log 20{,}347 = 4.3085.$$

(3) Find the logarithm of 0.0005076.

In the line with 50, and in the 7 column, is seen 7050; in the 8 column, 7059; the difference is 9. The difference between 5070 and 5080 is 10, and the difference between 5070 and 5076 is 6. Hence, $\frac{6}{10}$ of $9 = 5$ must be added to 7050. That is,

$$\log 0.0005076 = 6.7055 - 10.$$

315. To find a Number when its Logarithm is given.

(1) Find the number of which the logarithm is 1.9736.

Look for 9736 in the table. In the column headed "N," and in the line with 9736, is seen 94, and at the head of the column in which 9736 stands is seen 1. Therefore, write 941 and insert the decimal point as the characteristic directs. That is, the number required is 94.1.

(2) Find the number of which the logarithm is 3.7936.

Look for 7936 in the table. It cannot be found, but the two adjacent mantissas between which it lies are 7931 and 7938; their difference is 7, and the difference between 7931 and 7936 is 5. Therefore, $\frac{5}{7}$ of the difference between the numbers corresponding to the mantissas, 7931 and 7938, must be added to the number corresponding to the mantissa 7931.

The number corresponding to the mantissa 7938 is 6220.

The number corresponding to the mantissa 7931 is 6210.

The difference between these numbers is 10, and

$$6210 + \tfrac{5}{7} \text{ of } 10 = 6217.$$

Therefore, the number required is 6217.

(3) Find the number of which the logarithm is 7.3882 − 10.

Look for 3882 in the table. It cannot be found, but the two adjacent mantissas between which it lies are 3874 and 3892; their difference is 18, and the difference between 3874 and 3882 is 8. Therefore, $\frac{8}{18}$ of the difference between the numbers corresponding to the mantissas, 3874 and 3892, must be added to the number corresponding to the mantissa 3874.

The number corresponding to the mantissa 3892 is 2450.

The number corresponding to the mantissa 3874 is 2440.

The difference between these numbers is 10, and

$$2440 + \tfrac{8}{18} \text{ of } 10 = 2444.$$

Therefore, the number required is 0.002444.

(4) Find the number of which the logarithm is 0.3664.

The two adjacent mantissas between which the given mantissa 3664 lies are 3655 and 3674; their difference is 19, and the difference between 3655 and 3664 is 9.

The number corresponding to the mantissa 3655 is 2320.

Therefore, the number required is $2.320 + \frac{9}{19}$ of $10 = 2.325$.

Exercise 45

Find, from the table, the common logarithm of:

1. 60.	4. 3780.	7. 70,633.	10. 0.0004523.
2. 101.	5. 5432.	8. 12,028.	11. 0.01342.
3. 999.	6. 9081.	9. 0.00987.	12. 0.19873.

Find antilogarithms to the following common logarithms:

13. 4.2488.	16. 1.9730.	19. 9.0410 – 10.
14. 3.6330.	17. 0.1728.	20. 9.8420 – 10.
15. 4.7317.	18. 2.7635.	21. 7.7423 – 10.

Find the cologarithm of:

22. 428.	25. 4872.	28. 62,784.	31. 0.14964.
23. 567.	26. 9645.	29. 18,657.	32. 0.000762.
24. 841.	27. 0.478.	30. 0.00634.	33. 0.01783.

316. Computation by Logarithms.

(1) Find the product of $908.4 \times 0.05392 \times 2.117$.

$$\begin{aligned}
\log\ 908.4 &= 2.9583 \\
\log 0.05392 &= 8.7318 - 10 \\
\log\ 2.117 &= \underline{0.3257\qquad} \\
& \ 2.0158 \qquad = \log 103.7.
\end{aligned}$$

Therefore, the required product is 103.7.

When any factor is *negative* find its logarithm without regard to the sign; write n after the logarithm that corresponds to a negative number. If the number of logarithms so marked is *odd*, the product is *negative*; if *even*, the product is *positive*.

(2) Find the product of $4.52 \times (-0.3721) \times 0.912$.

$$\begin{aligned}
\log\ 4.52 &= 0.6551 \\
\log 0.3721 &= 9.5706 - 10\,n \\
\log\ 0.912 &= \underline{9.9600 - 10\ } \\
& \ 0.1857\,n \qquad = \log -1.534.
\end{aligned}$$

Therefore, the required product is -1.534.

(3) Find the cube of 0.0497.

$$\begin{array}{rl} \log\ 0.0497 = & 8.6964 - 10 \\ & \underline{\qquad\quad 3 \qquad\quad} \\ & 6.0892 - 10 = \log 0.0001228. \end{array}$$

Therefore, the cube of 0.0497 is 0.0001228.

(4) Find the fourth root of 0.00862.

$$\begin{array}{rl} \log 0.00862 = & 7.9355 - 10 \\ & \underline{30. \qquad - 30} \\ 4\,\big) & 37.9355 - 40 \\ & 9.4839 - 10 = \log 0.3047. \end{array}$$

Therefore, the fourth root of 0.00862 is 0.3047.

(5) Find the value of $\sqrt[5]{\dfrac{3.1416 \times 4771.2 \times 2.718^{\frac{1}{3}}}{30.13^4 \times 0.4343^{\frac{1}{2}} \times 69.89^4}}$.

$$\begin{array}{rlrl} \log 3.1416 & = 0.4971 & = & 0.4971 \\ \log 4771.2 & = 3.6786 & = & 3.6786 \\ \tfrac{1}{3} \log 2.718 & = \tfrac{1}{3}(0.4343) & = & 0.1448 \\ 4 \operatorname{colog} 30.13 & = 4\,(8.5210 - 10) & = & 4.0840 - 10 \\ \tfrac{1}{2} \operatorname{colog} 0.4343 & = \tfrac{1}{2}(0.3622) & = & 0.1811 \\ 4 \operatorname{colog} 69.89 & = 4\,(8.1556 - 10) & = & \underline{2.6224 - 10} \\ & & & 11.2080 - 20 \\ & & & \underline{30. \qquad - 30} \\ & & 5\,\big) & 41.2080 - 50 \\ & & & 8.2416 - 10 = \log 0.01744. \end{array}$$

Therefore, the required value is 0.01744.

317. An **exponential equation**, that is, an equation in which the exponent involves the unknown number, is easily solved by logarithms.

Find the value of x in $81^x = 10$.

$$\begin{aligned} 81^x &= 10. \\ \therefore \log (81^x) &= \log 10, \\ x \log 81 &= \log 10, \\ x &= \frac{\log 10}{\log 81} = \frac{1.0000}{1.9085} = 0.524. \end{aligned}$$

Exercise 46

Find by logarithms:

1. 948.76×0.043875.
2. 3.4097×0.0087634.
3. 830.75×0.0003769.
4. 8.4395×0.98274.
5. $7564 \times (-0.003764)$.
6. $3.765 \times (-0.08345)$.
7. $-5.8404 \times (-0.00178)$.
8. -8945×73.85.
9. $\dfrac{70654}{54013}$.
10. $\dfrac{7.652}{-0.06875}$.
11. $\dfrac{0.07654}{83.947 \times 0.8395}$.
12. $\dfrac{212 \times (-6.12) \times (-2008)}{365 \times (-531) \times 2.576}$.
13. 1.1768^5.
14. 1.3178^{10}.
15. $11^{\frac{1}{5}}$.
16. $(\frac{73}{81})^{11}$.
17. $(\frac{14}{51})^7$.
18. $906.80^{\frac{1}{4}}$.
19. $(\frac{951}{823})^6$.
20. $(7\frac{6}{11})^{0.38}$.
21. $2.5637^{\frac{3}{11}}$.
22. $(8\frac{3}{4})^{2.3}$.
23. $(5\frac{31}{37})^{0.375}$.
24. $(9\frac{20}{43})^{\frac{1}{3}}$.
25. $\sqrt[5]{0.00476}$.
26. $\sqrt[3]{-325}$.
27. $(-400)^{\frac{2}{3}}$.
28. $(0.00065)^{\frac{2}{5}}$.
29. $(-0.0084)^{\frac{2}{3}}$.
30. $(0.00872)^{\frac{4}{7}}$.
31. $(0.8756)^{\frac{1}{5}}$.
32. $(-0.4762)^{\frac{3}{7}}$.
33. $\sqrt[7]{8462}$.
34. $\sqrt[5]{0.481}$.
35. $(-286)^{\frac{2}{5}}$.
36. $(-4762)^{\frac{3}{11}}$.
37. $(4.861)^{\frac{1}{6}}$.
38. $(-0.00222)^{\frac{4}{7}}$.
39. $(-0.03654)^{\frac{2}{5}}$.
40. $(-0.00008)^{\frac{1}{5}}$.
41. $\dfrac{(-4)^{\frac{2}{7}}}{2^{\frac{1}{3}}}$.
42. $(\frac{2}{5})^{\frac{1}{2}}$.
43. $\dfrac{\sqrt[3]{0.00052}}{\sqrt[5]{0.0068125}}$.
44. $\dfrac{4\,(0.6235)^{\frac{1}{2}}}{(-257.14)^{\frac{1}{3}}}$.

45. $\sqrt[4]{\dfrac{0.008541^2 \times 8641 \times 4.276^{\frac{1}{3}} \times 0.0084}{0.00854^3 \times 182.63^{\frac{1}{4}} \times 82^{\frac{1}{3}} \times 487.27^{\frac{1}{2}}}}.$

46. $\sqrt[5]{\dfrac{0.0075433^2 \times 78.343 \times 8172.4^{\frac{1}{3}} \times 0.00052}{64285^{\frac{1}{3}} \times 154.27^4 \times 0.001 \times 586.79^{\frac{1}{2}}}}.$

47. $\sqrt[7]{\dfrac{0.03271^2 \times 53.429 \times 0.77542^6}{32.769 \times 0.000371^4}}.$

48. $\sqrt[3]{\dfrac{7.1206 \times \sqrt{0.13274} \times 0.057389}{\sqrt{0.43468} \times 17.385 \times \sqrt{0.0096372}}}.$

Find x from the equations:

49. $5^x = 12.$
50. $4^x = 40.$
51. $7^x = 25.$
52. $(1.3)^x = 7.2.$
53. $(0.4)^{-x} = 7.$
54. $(0.9)^{\frac{1}{x^2}} = (4.7)^{-\frac{1}{2}}.$

318. Change of System. Logarithms to any base a may be converted into logarithms to any other base b as follows:

Let N be any number, and let

$$n = \log_a N \text{ and } m = \log_b N.$$

Then, $$N = a^n \text{ and } N = b^m.$$

$$\therefore a^n = b^m.$$

Taking logarithms to any base r,

$$n \log_r a = m \log_r b, \qquad (\S\ 300)$$

or, $$\log_r a \times \log_a N = \log_r b \times \log_b N,$$

from which $\log_b N$ may be found when $\log_r a$, $\log_r b$, and $\log_a N$ are given; and conversely, $\log_a N$ may be found when $\log_r a$, $\log_r b$, and $\log_b N$ are given.

319. If $a = 10$, $b = e$, $r = 10$, and $N = 10$,

$$\log_{10} 10 \times \log_{10} 10 = \log_{10} e \times \log_e 10. \qquad (\S\ 318)$$

$$\therefore \log_e 10 = \frac{1}{\log_{10} e}.$$

From tables, $\log_{10} e = 0.4342945.$

$$\therefore \log_e 10 = 2.3025851.$$

320. If $a = 10$, $b = e$, $r = 10$, and N is any number,

$$\log_{10} 10 \times \log_{10} N = \log_{10} e \times \log_e N. \qquad (\S\ 318)$$

$$\therefore \log_e N = \frac{1}{\log_{10} e} \times \log_{10} N,$$

and $$\log_{10} N = \log_{10} e \times \log_e N.$$

Hence, to convert common logarithms into natural logarithms, multiply by 2.3025851; and to convert natural logarithms into common logarithms, multiply by 0.4342945.

Exercise 47

Find to four digits the natural logarithm of:

1. 2.	3. 100.	5. 7.89.	7. 2.001.
2. 3.	4. 32.5.	6. 1.23.	8. 0.0931.

Find to four digits:

9. $\log_2 7$.	11. $\log_4 9$.	13. $\log_9 8$.	15. $\log_7 14$.
10. $\log_3 4$.	12. $\log_5 7$.	14. $\log_8 5$.	16. $\log_5 102$.

17. Find the logarithm of 4 in the system of which $\frac{1}{3}$ is the base.

18. Find the logarithm of $\frac{7}{11}$ in the system of which 0.5 is the base.

19. Find the base of the system in which the logarithm of 8 is $\frac{2}{3}$.

20. Find the base of the system in which the logarithm of $\frac{2}{3}$ is $-\frac{3}{5}$.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

CHAPTER XXI

INTEREST AND ANNUITIES

321. Interest is money paid for the use of money.

322. Principal. The sum loaned is the *principal.*

323. Rate of Interest. The rate of interest is the interest on $1 for one year.

324. Amount. The sum of the principal and interest is the *amount.*

325. Compound Interest. Interest is *compounded* when it is added to the principal and becomes a part of the principal at specified intervals.

Compound interest is compounded annually, semiannually, quarterly, or monthly according to agreement. Compound interest is understood to be compounded annually unless otherwise stated.

326. In **interest problems** four elements are considered: *principal, rate, time,* and *interest* or *amount.* If three of the elements are known, the fourth may be found.

327. Let r stand for the interest on $1 for one year; t for the time in years between two successive conversions (compoundings); n the number of conversions; A_0 the original amount, the principal; A_n the amount after n conversions of interest into principal; and I_n the total of the interest converted in the n conversions. Then,

$$A_1 = A_0(1 + rt),$$

$$A_2 = A_1(1 + rt) = A_0(1 + rt)^2,$$

$$A_3 = A_2(1 + rt) = A_0(1 + rt)^3,$$

$$A_4 = A_3(1 + rt) = A_0(1 + rt)^4,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$A_n = A_{n-1}(1 + rt) = A_0(1 + rt)^n,$$

and $$I_n = A_n - A_0.$$

If R is written for $(1 + rt)$, these equations become

$$\mathbf{A_n = A_0 R^n}, \qquad [1]$$

and $$\mathbf{I_n = A_0 (R^n - 1)}. \qquad [2]$$

Hence, also, $\log A_n = \log A_0 + n \log(1 + rt)$.

In the case of simple interest, n = 1.

328. If there should occur a *broken* period whose time in years is t', t' being less than t, the rate of increase for t' is by commercial usage taken to be $1 + rt'$.

329. Sinking Funds. If the sum set apart at the end of each year to be put at compound interest is represented by S,

The sum at the end of the

first year $= S$,
second year $= S + SR$,
third year $= S + SR + SR^2$,
nth year $= S + SR + SR^2 + \cdots + SR^{n-1}$.

That is, the amount $A = S + SR + SR^2 + \cdots + SR^{n-1}$.

$$\therefore AR = SR + SR^2 + SR^3 + \cdots + SR^n.$$

$$\therefore AR - A = SR^n - S.$$

$$\therefore A = \frac{S(R^n - 1)}{R - 1},$$

or, $$\mathbf{A = \frac{S(R^n - 1)}{r}}.$$

(1) If \$10,000 is set apart annually, and put at 6 per cent compound interest for 10 years, what will be the amount?

$$A = \frac{S(R^n - 1)}{r} = \frac{\$10,000(1.06^{10} - 1)}{0.06}.$$

By four-place logarithms the amount is \$131,600.

(2) A county owes \$60,000. What sum must be set apart annually, as a sinking fund, to cancel the debt in 10 years, provided money is worth 6 per cent?

$$S = \frac{Ar}{R^n - 1} = \frac{\$60,000 \times 0.06}{1.06^{10} - 1} = \$4558 \text{ (by four-place logs.)}.$$

NOTE. The amount of tax required yearly is \$3600 for the *interest* and \$4558 for the sinking fund; that is, \$8158.

330. Annuities. A sum of money that is payable yearly, or in parts at fixed periods in the year, is called an **annuity.**

To find the amount of an unpaid annuity when the interest, time, and rate per cent are given.

The sum due at the *end* of the

$$\begin{aligned} &\text{first year} &&= S, \\ &\text{second year} &&= S + SR, \\ &\text{third year} &&= S + SR + SR^2, \\ &n\text{th year} &&= S + SR + SR^2 + \cdots + SR^{n-1}. \end{aligned}$$

That is, $$\mathbf{A} = \frac{\mathbf{S}(\mathbf{R}^n - 1)}{\mathbf{r}}. \qquad (\S\ 329)$$

An annuity of \$1200 was unpaid for 6 years. What was the amount due if interest is reckoned at 6 per cent?

$$A = \frac{S(R^n - 1)}{r} = \frac{\$1200(1.06^6 - 1)}{0.06} = \$8360 \text{ (by four-place logs.)}.$$

331. *To find the present worth of an annuity when the time it is to continue and the rate per cent are given.*

Let P denote the present worth. Then, the amount of P for n years is equal to A, the amount of the annuity for n years.

But the amount of P for n years

$$= P(1 + r)^n = PR^n, \qquad (\S\ 327)$$

and $$A = \frac{S(R^n - 1)}{R - 1}. \qquad (\S\ 330)$$

$$\therefore PR^n = \frac{S(R^n - 1)}{R - 1}.$$

$$\therefore \mathbf{P} = \frac{\mathbf{S}}{\mathbf{R^n}} \times \frac{\mathbf{R^n - 1}}{\mathbf{R - 1}}.$$

This equation may be written

$$P = \frac{S}{R-1} \times \frac{R^n - 1}{R^n} = \frac{S}{R-1}\left(1 - \frac{1}{R^n}\right).$$

As n increases indefinitely, the expression $1 - \frac{1}{R^n}$ approximates to 1.

Therefore, if the annuity is *perpetual,*

$$P = \frac{S}{R-1} = \frac{S}{r}.$$

(1) Find the present worth of an annual pension of \$105, for 5 years, at 4 per cent interest.

$$P = \frac{S}{R^n} \times \frac{R^n - 1}{R - 1} = \frac{\$105}{1.04^5} \times \frac{1.04^5 - 1}{1.04 - 1} = \$466.20 \text{ (by logs.)}.$$

(2) Find the present worth of a perpetual scholarship that pays \$300 annually, at 6 per cent interest.

$$P = \frac{S}{r} = \frac{\$300}{0.06} = \$5000.$$

332. *To find the present worth of an annuity that begins in a given number of years, when the time it is to continue and the rate per cent are given.*

Let p denote the number of years before the annuity begins, and q the number of years the annuity is to continue.

Then, the present worth of the annuity to the time it *terminates* is

$$\frac{S}{R^{p+q}} \times \frac{R^{p+q} - 1}{R - 1}. \qquad (\S\ 331)$$

The present worth of the annuity to the time it *begins* is

$$\frac{S}{R^p} \times \frac{R^p - 1}{R - 1}. \qquad (\S\ 331)$$

Hence,
$$P = \left(\frac{S}{R^{p+q}} \times \frac{R^{p+q} - 1}{R - 1}\right) - \left(\frac{S}{R^p} \times \frac{R^p - 1}{R - 1}\right)$$
$$= \frac{S}{R^{p+q}}\left[\frac{R^{p+q} - 1}{R - 1} - \frac{R^q(R^p - 1)}{R - 1}\right]$$
$$= \frac{S}{R^{p+q}}\left[\frac{R^{p+q} - 1 - R^{p+q} + R^q}{R - 1}\right].$$
$$\therefore \mathbf{P} = \frac{\mathbf{S}}{\mathbf{R}^{p+q}} \times \frac{\mathbf{R}^q - 1}{\mathbf{R} - 1}.$$

If the annuity is to begin at the end of p years, and to be perpetual, the formula

$$P = \frac{S}{R^{p+q}} \times \frac{R^q - 1}{R - 1}$$

may be written
$$P = \frac{S}{R^p(R - 1)} \times \frac{R^q - 1}{R^q}.$$

Since $\frac{R^q - 1}{R^q}$ approaches 1 as q increases indefinitely (§ 331),

$$P = \frac{S}{R^p(R - 1)}.$$

(1) Find the present worth of an annuity of \$5000, to begin in 6 years, and to continue 12 years, at 6 per cent interest.

$$P = \frac{S}{R^{p+q}} \times \frac{R^q - 1}{R - 1} = \frac{\$5000}{1.06^{18}} \times \frac{1.06^{12} - 1}{0.06} = \$29{,}550 \text{ (by logs.)}.$$

(2) Find the present worth of a perpetual annuity of \$1000, to begin in 3 years, at 4 per cent interest.

$$P = \frac{S}{R^p(R - 1)} = \frac{\$1000}{1.04^3 \times 0.04} = \$22{,}225 \text{ (by logs.)}.$$

333. *To find the annuity when the present worth, the time, and the rate per cent are given.*

$$P = \frac{S(R^n - 1)}{R^n(R - 1)}. \qquad (\S\ 331)$$

$$\therefore S = \frac{PR^n(R - 1)}{R^n - 1}.$$

$$\therefore \mathbf{S = Pr \times \frac{R^n}{R^n - 1}}.$$

What annuity for 5 years will \$4675 yield when interest is reckoned at 4 per cent?

$$S = Pr \times \frac{R^n}{R^n - 1} = \$4675 \times 0.04 \times \frac{1.04^5}{1.04^5 - 1} = \$1053 \text{ (by logs.)}.$$

334. Life Insurance. In order that a certain sum may be secured, to be payable at his death, a person pays yearly a fixed *premium.*

If P denotes the premium to be paid for n years to insure an amount A, to be paid immediately after the last premium, then

$$A = \frac{P(R^n - 1)}{R - 1}. \qquad (\S\ 329)$$

$$\therefore \mathbf{P = \frac{A(R - 1)}{R^n - 1} = \frac{Ar}{R^n - 1}}.$$

If A is to be paid a year after the last premium, then

$$\mathbf{P = \frac{A(R - 1)}{R(R^n - 1)} = \frac{Ar}{R(R^n - 1)}}.$$

NOTE. In the calculation of life insurance it is necessary to employ tables that show for every age the *probable duration of life.*

335. Bonds. If P denotes the price of a bond that has n years to run, and bears r per cent interest, S the face of the bond, and q the current rate of interest, what interest on his investment will a purchaser of such a bond receive?

Let x denote the rate of interest on the investment.

Then $P(1 + x)^n$ is the value of the purchase money at the end of n years.

$Sr(1+q)^{n-1}+Sr(1+q)^{n-2}+\cdots+Sr+S$ is the amount received on the bond if the interest received from the bond is put immediately at compound interest at q per cent.

But $Sr(1+q)^{n-1}+Sr(1+q)^{n-2}+\cdots+Sr$ is a geometrical progression in which the first term is Sr, the ratio $1+q$, and the number of terms n.

Therefore, $Sr(1+q)^{n-1}+Sr(1+q)^{n-2}+\cdots+Sr+S$

$$= S + \frac{Sr[(1+q)^n - 1]}{q}. \qquad (\S\ 276)$$

$$\therefore P(1+x)^n = S + \frac{Sr[(1+q)^n - 1]}{q}.$$

$$\therefore 1+x = \left[\frac{S}{P} + \frac{Sr[(1+q)^n - 1]}{Pq}\right]^{\frac{1}{n}}.$$

$$\therefore 1+x = \left[\frac{Sq + Sr(1+q)^n - Sr}{Pq}\right]^{\frac{1}{n}}.$$

(1) What interest will a purchaser receive on his investment if he buys at 114 a 4 per cent bond that has 26 years to run, money being worth $3\frac{1}{2}$ per cent?

$$1+x = \left(\frac{3.5 + 4 \times 1.035^{26} - 4}{114 \times 0.035}\right)^{\frac{1}{26}}.$$

By logarithms, $1+x = 1.033$.

That is, the purchaser will receive $3\frac{1}{3}$ per cent for his money.

(2) At what price must 7 per cent bonds, running 12 years, with the interest payable semiannually, be bought in order that the purchaser may receive on his investment 5 per cent, interest semiannual, which is the current rate of interest?

$$P = \frac{Sq + Sr(1+q)^n - Sr}{q(1+x)^n}.$$

In this case $S = 100$; and, as the interest is semiannual,

$$q = 0.025,\ r = 0.035,\ n = 24,\ x = 0.025.$$

Then,
$$P = \frac{2.5 + 3.5(1.025)^{24} - 3.5}{0.025(1.025)^{24}}.$$

By logarithms,
$$P = 118.$$

Exercise 48

1. In how many years will \$100 amount to \$1050 at 5 per cent compound interest?

2. In how many years will \$$A$ amount to \$$B$ (1) at simple interest, (2) at compound interest, r and R being used in their usual sense?

3. Find the difference (to five places of decimals) between the amount of \$1 in 2 years, at 6 per cent compound interest, according as the interest is payable yearly or monthly.

4. At 5 per cent, find the amount of an annuity of \$$A$ which has been left unpaid for 4 years.

5. Find the present value of an annuity of \$100 for 5 years, reckoning interest at 4 per cent.

6. A perpetual annuity of \$1000 is to be purchased, to begin at the end of 10 years. If interest is reckoned at $3\frac{1}{2}$ per cent, what should be paid for the annuity?

7. A debt of \$1850 is discharged by two payments of \$1000 each, at the end of one and two years. Find the rate of interest paid.

8. Reckoning interest at 4 per cent, what annual premium should be paid for 30 years in order to secure \$2000 to be paid at the end of that time, the premium being due at the beginning of each year?

9. An annual premium of \$150 is paid to a life-insurance company for insuring \$5000. If money is worth 4 per cent, for how many years must the premium be paid in order that the company may sustain no loss?

10. What may be paid for bonds due in 10 years, and bearing semiannual coupons of 4 per cent each, in order to realize 3 per cent semiannually, if money is worth 3 per cent semiannually?

11. When money is worth 2 per cent semiannually, if bonds having 12 years to run, and bearing semiannual coupons of $3\frac{1}{2}$ per cent each, are bought at $114\frac{1}{8}$, what per cent is realized on the investment?

12. If \$126 is paid for bonds due in 12 years, and yielding $3\frac{1}{2}$ per cent semiannually, what per cent is realized on the investment, provided money is worth 2 per cent semiannually?

13. A person borrows \$600.25. How much must he pay annually that the whole debt may be discharged in 35 years, allowing simple interest at 4 per cent?

14. A perpetual annuity of \$100 a year is sold for \$2500. At what rate is the interest reckoned?

15. A perpetual annuity of \$320, to begin 10 years hence, is to be purchased. If interest is reckoned at $3\frac{1}{2}$ per cent, what should be paid for the annuity?

16. A sum of \$10,000 is loaned at 4 per cent. At the end of the first year a payment of \$400 is made, and at the end of each following year a payment is made greater by 30 per cent than the preceding payment. Find in how many years the debt will be paid.

17. A man with a capital of \$100,000 spends every year \$9000. If the current rate of interest is 5 per cent, in how many years will he be ruined?

18. Find the amount of \$365 at compound interest for 20 years, at 5 per cent.

19. A railroad company bought and paid for 850 freight cars at \$360 each. The company wishes to charge the cost of the cars to operating expenses in six equal annual amounts, the first charge to be made on the date of the purchase. If money is worth 4%, what annual charge to operating expenses should be made?

CHAPTER XXII

CHOICE

336. Fundamental Principle. *If one thing can be done in* a *different ways, and, when it has been done, a second thing can be done in* b *different ways, then the two things can be done together in* a × b *different ways.*

For, corresponding to the *first* way of doing the first thing, there are b different ways of doing the second thing; corresponding to the *second* way of doing the first thing, there are b different ways of doing the second thing; and so on for *each* of the a different ways of doing the first thing.

Therefore, there are $a \times b$ different ways of doing the two things together.

(1) If a box contains four capital letters, A, B, C, D, and three small letters, x, y, z, in how many different ways may two letters, one a capital letter and one a small letter, be selected?

A capital letter may be selected in four different ways, since any one of the letters A, B, C, D may be selected. A small letter may be selected in three different ways, since any one of the letters x, y, z may be selected. Any small letter may be put with any capital letter.

Thus, with A we may put x, or y, or z;
with B we may put x, or y, or z;
with C we may put x, or y, or z;
with D we may put x, or y, or z.

Hence, the number of ways in which a selection may be made is 4×3, or 12. These ways are:

Ax	Bx	Cx	Dx
Ay	By	Cy	Dy
Az	Bz	Cz	Dz

(2) On a shelf are 7 English, 5 French, and 9 German books. In how many different ways may two books, not in the same language, be selected?

An English book and a French book can be selected in 7×5, or 35, ways. A French book and a German book in 5×9, or 45, ways. An English book and a German book in 7×9, or 63, ways.

Hence, there is a choice of $35 + 45 + 63$, or 143, ways.

(3) Out of the ten figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 how many different numbers of two figures each can be formed?

Since 0 has no value in the left-hand place, the left-hand place can be filled in 9 ways.

The right-hand place can be filled in 10 ways, since *repetitions* of the digits are allowed, as in 22, 33, etc.

Hence, the whole number of numbers is 9×10, or 90.

337. By successive application of the principle of § 336 it may be shown that,

If one thing can be done in a *different ways, and then a second thing can be done in* b *different ways, then a third thing in* c *different ways, then a fourth thing in* d *different ways, and so on, the number of different ways of doing all the things together is* $a \times b \times c \times d \times \cdots$

For, the first and second things can be done together in $a \times b$ different ways (§ 336), and the third thing in c different ways; hence, by § 336, the first and second things and the third thing can be done together in $(a \times b) \times c$ different ways. Therefore, the first three things can be done in $a \times b \times c$ different ways. And so on, for any number of things.

In how many different ways can four Christmas presents be given to four boys, one to each boy?

The first present may be given to any one of the boys; hence, there are 4 ways of disposing of it.

When the first present has been disposed of, the second present may be given to any one of the other three boys; hence, there are 3 ways of disposing of it.

When the first and second presents have been disposed of, the third present may be given to either of the two other boys; hence, there are 2 ways of disposing of it.

When the first, second, and third presents have been disposed of, the fourth present must be given to the last boy; hence, there is only 1 way of disposing of it.

There are, then, $4 \times 3 \times 2 \times 1$, or 24, ways.

338. Combinations and Permutations. (1) In how many different ways can a vowel and a consonant be chosen, assuming that the alphabet contains 6 vowels and 20 consonants?

A vowel can be chosen in 6 ways and a consonant in 20 ways, and both (§ 336) in 6×20, or 120, ways.

(2) In how many different ways can a two-lettered word be made, containing one vowel and one consonant?

The vowel can be chosen in 6 ways and the consonant in 20 ways; and then each combination of a vowel and a consonant can be written in 2 ways; as, *ac*, *ca*.

Hence, the whole number of ways is $6 \times 20 \times 2$, or 240.

These two examples show the difference between a *selection*, or *combination*, of different things and an *arrangement*, or *permutation*, of the same things.

Thus, *ac* forms a selection of a vowel and a consonant, and *ac* and *ca* form two different *arrangements* of this selection.

From (1) it is seen that 120 different selections can be made with a vowel and a consonant; and from (2) it is seen that 240 different *arrangements* can be made with these selections.

Again, a, b, c is a selection of three letters from the alphabet. This selection admits of 6 different arrangements, as follows:

$$\begin{array}{ccc} abc & bca & cab \\ acb & bac & cba \end{array}$$

A **selection**, or **combination**, of any number of things is a group of that number of things put together without regard to their order.

An **arrangement**, or **permutation**, of any number of things is a group of that number of things put together in a definite order.

339. Permutations, Things all Different. *The number of different arrangements or permutations of* n *different things taken all together is*

$$n(n-1)(n-2)(n-3)\times\cdots\times 3\times 2\times 1.$$

For, the first place can be filled in n ways, then the second place in $n-1$ ways, then the third place in $n-2$ ways, and so on, to the last place, which can be filled in only 1 way.

Hence (§ 337), the whole number of arrangements is the continued product,

$$n(n-1)(n-2)(n-3)\times\cdots\times 3\times 2\times 1.$$

For the sake of brevity this product is often written $\underline{|n}$ or $n!$ (read *factorial* n).

Observe that $1\times 2\times\cdots\times(n-1)\times n=\underline{|n}$.

How many different arrangements of nine letters each can be formed with the letters in the word *Cambridge?*

There are nine letters. In making any arrangement any one of the letters can be put in the first place. Hence, the first place can be filled in 9 ways.

Then, the second place can be filled with any one of the remaining eight letters; that is, in 8 ways.

In like manner, the third place can be filled in 7 ways, the fourth place in 6 ways, and so on; and, lastly, the ninth place in 1 way.

If the nine places are indicated by Roman numerals, the result is (§ 337) as follows:

$$\begin{array}{ccccccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} & \text{VI} & \text{VII} & \text{VIII} & \text{IX} \\ 9\times & 8\times & 7\times & 6\times & 5\times & 4\times & 3\times & 2\times & 1 = 362{,}880 \text{ ways.} \end{array}$$

Hence, there are 362,880 different arrangements possible.

340. *The number of different permutations of* n *different things taken* r *at a time is*

$$n(n-1)(n-2)\cdots \text{to } r \text{ factors,}$$

that is, $$n(n-1)(n-2)\cdots[n-(r-1)],$$

or $$n(n-1)(n-2)\cdots(n-r+1).$$

For, the first place can be filled in n ways, then the second place in $n-1$ ways, then the third place in $n-2$ ways, and so on, and then the rth place in $n-(r-1)$ ways.

Let $P_{n,r}$ represent the number of arrangements of n different things taken r at a time. Then,

$$P_{n,r} = n(n-1)(n-2)\cdots \text{to } r \text{ factors}$$
$$= n(n-1)(n-2)\cdots(n-r+1).$$

How many different arrangements of four letters each can be formed from the letters in the word *Cambridge?*

There are nine letters and four places to be filled.

The first place can be filled in 9 ways. Then, the second place can be filled in 8 ways; then, the third place in 7 ways; and then, the fourth place in 6 ways.

If the places are indicated by I, II, III, IV, the result is (§ 337)

$$\begin{array}{l} \text{I} \quad \text{II} \quad \text{III} \quad \text{IV} \\ 9 \times 8 \times 7 \times 6 = 3024 \text{ ways.} \end{array}$$

Hence, there are 3024 different arrangements possible.

341. Combinations, Things all Different. *The number of different selections or combinations of* n *different things taken* r *at a time is*

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}}.$$

To prove this, let $C_{n,r}$ represent the number of different selections or combinations of n different things taken r at a time.

Take one selection of r things; from this selection $\underline{|r}$ arrangements can be made (§ 339).

Take a second selection; from this selection $\underline{|r}$ arrangements can be made. And so on, for *each* of the $C_{n,r}$ selections.

Hence, $C_{n,r} \times \underline{|r}$ is the number of *arrangements* of n different things taken r at a time.

That is, $$C_{n,r} \times \underline{|r} = P_{n,r}.$$

$$\therefore C_{n,r} = \frac{P_{n,r}}{\lfloor r}.$$

$$\therefore C_{n,r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r}.$$

In how many different ways can three vowels be selected from the five vowels *a, e, i, o, u?*

The number of different ways in which we can *arrange* 3 vowels out of 5 is (§ 340) $5 \times 4 \times 3$, or 60.

These 60 arrangements might be obtained by first forming all the possible selections of 3 vowels out of 5, and then arranging the 3 vowels in each selection in as many ways as possible.

Since each selection can be arranged in $\lfloor 3$, or 6, ways (§ 339), the number of selections is $\frac{60}{6}$, or 10.

The formula applied to this problem gives

$$C_{5,3} = \frac{5 \times 4 \times 3}{1 \times 2 \times 3} = 10.$$

342. Combinations, Second Formula. Multiplying both numerator and denominator of the expression for the number of combinations in the last example by 2×1, we have

$$C_{5,3} = \frac{5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 2 \times 1} = \frac{\lfloor 5}{\lfloor 3 \lfloor 2}.$$

In general, multiplying both numerator and denominator of the expression for $C_{n,r}$ in § 341 by $\lfloor n-r$, we have

$$C_{n,r} = \frac{n(n-1)\cdots(n-r+1)(n-r)\cdots 1}{\lfloor r \times (n-r)\cdots 1} = \frac{\lfloor n}{\lfloor r \lfloor n-r}.$$

This second form is more compact than the first and is more easily remembered.

NOTE. In the reduction of such a result $\lfloor n-r$ cancels all the factors of the numerator from 1 up to and including $n-r$. Thus, in $\frac{\lfloor 12}{\lfloor 5 \lfloor 7}$, $\lfloor 7$ cancels all the factors of $\lfloor 12$ from 1 up to and including 7; so that

$$\frac{\lfloor 12}{\lfloor 5 \lfloor 7} = \frac{12 \times 11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4 \times 5} = 792.$$

343. Theorem. *The number of combinations of* n *things taken* r *at a time is the same as the number of combinations of* n *things taken* n − r *at a time.*

For, $C_{n, n-r} = \dfrac{\lfloor n}{\lfloor n-r \lfloor n-(n-r)} = \dfrac{\lfloor n}{\lfloor n-r \lfloor r} = C_{n, r}.$

This is also evident from the fact that for every selection of r things *taken*, a selection of $n - r$ things is *left*.

Thus, out of 8 things, 3 things can be selected in the same number of ways as 5 things; namely,

$$\frac{\lfloor 8}{\lfloor 3 \lfloor 5} = \frac{8 \times 7 \times 6}{1 \times 2 \times 3} = 56 \text{ ways.}$$

Out of 10 things, 7 things can be selected in the same number of ways as 3 things; namely,

$$\frac{\lfloor 10}{\lfloor 7 \lfloor 3} = \frac{10 \times 9 \times 8}{1 \times 2 \times 3} = 120.$$

344. Examples in Combinations and Permutations. Of the permutations possible with the letters of the word *Cambridge*, taken all together:

(1) How many begin with a vowel?

In filling the nine places of any arrangement the first place can be filled in only 3 ways, the other places in $\lfloor 8$ ways.

Hence, the answer is $3 \times \lfloor 8 = 120{,}960$. (§ 337)

(2) How many both begin and end with a vowel?

The first place can be filled in 3 ways, the last place in 2 ways (one vowel having been used), and the remaining seven places in $\lfloor 7$ ways.

Hence, the answer is $3 \times 2 \times \lfloor 7 = 30{,}240$. (§ 337)

(3) How many begin with *Cam*?

The answer is evidently $\lfloor 6$, since our only choice lies in arranging the remaining six letters of the word.

(4) How many have the letters *c a m* standing together?

This may be resolved into arranging the group *c a m* and the last six letters, regarded as seven distinct elements, and then arranging the letters *c a m*.

The first can be done in $\underline{|7}$ ways, and the second in $\underline{|3}$ ways. Hence, both can be done in $\underline{|7} \times \underline{|3} = 30{,}240$ ways.

In how many ways can the letters of the word *Cambridge* be written:

(5) Without changing the *place* of any vowel?

The second, sixth, and ninth places can be filled each in only 1 way; the other places in $\underline{|6}$ ways.

Therefore, the whole number of ways is $\underline{|6} = 720$.

(6) Without changing the *order* of the 3 vowels?

The vowels in the different arrangements are to be kept in the order a, i, e.

One of the 6 consonants can be placed in 4 ways: *before a*, *between a* and *i*, *between i* and *e*, and *after e*.

Then, a second consonant can be placed in 5 ways, a third consonant in 6 ways, a fourth consonant in 7 ways, a fifth consonant in 8 ways, and the last consonant in 9 ways. Hence, the whole number of ways is

$$4 \times 5 \times 6 \times 7 \times 8 \times 9, \text{ or } 60{,}480.$$

(7) Out of 20 consonants, in how many ways can 18 be selected?

The number of ways in which the 18 can be selected is

$$\frac{\underline{|20}}{\underline{|18}\,\underline{|2}} = \frac{20 \times 19}{2} = 190. \qquad (\S\ 342)$$

(8) In how many ways can the same choice be made so as always to include the letter b?

Taking b first, we must then select 17 out of the remaining 19 consonants. This can be done in

$$\frac{\underline{|19}}{\underline{|17}\,\underline{|2}} = \frac{19 \times 18}{2} = 171 \text{ ways.} \qquad (\S\ 342)$$

(9) In how many ways can the same choice be made so as to include b and not include c?

Taking b first, we have then to choose 17 out of 18, c being excluded. This can be done in 18 ways.

(10) From 20 Republicans and 6 Democrats, in how many ways can 5 different offices be filled, of which 3 particular offices must be filled by Republicans, and the other 2 offices by Democrats?

The first 3 offices can be assigned to 3 Republicans in

$$20 \times 19 \times 18 = 6840 \text{ ways.}$$

The other 2 offices can be assigned to 2 Democrats in

$$6 \times 5 = 30 \text{ ways.}$$

There is, then, a choice of $6840 \times 30 = 205{,}200$ different ways.

(11) Out of 20 consonants and 6 vowels, in how many ways can we make a word consisting of 3 different consonants and 2 different vowels?

Three consonants can be selected in $\frac{20 \times 19 \times 18}{1 \times 2 \times 3} = 1140$ ways, and two vowels in $\frac{6 \times 5}{1 \times 2} = 15$ ways. Hence, the 5 letters can be selected in $1140 \times 15 = 17{,}100$ ways.

When 5 letters have been so selected they can be arranged in $\underline{|5} = 120$ different orders. Hence, there are $17{,}100 \times 120 = 2{,}052{,}000$ different ways of making the word.

Observe that the letters are first *selected* and then *arranged*.

(12) A society consists of 50 members, 10 of whom are physicians. In how many ways can a committee of 6 members be selected so as to include *at least* 1 physician?

Six members can be selected from the whole society in

$$\frac{\underline{|50}}{\underline{|6}\,\underline{|44}} \text{ ways.}$$

Six members can be selected from the whole society, so as to include *no physician*, by choosing them all from the 40 members who are not physicians, and this can be done in

$$\frac{\underline{|40}}{\underline{|6}\,\underline{|34}} \text{ ways.}$$

Hence, $\frac{\underline{|50}}{\underline{|6}\,\underline{|44}} - \frac{\underline{|40}}{\underline{|6}\,\underline{|34}}$ is the number of ways of selecting the committee so as to include at least 1 physician.

345. Greatest Number of Combinations. To find for what value of r the number of selections of n things, taken r at a time, is the greatest.

The formula

$$C_{n,r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1\times 2\times 3\times\cdots r}$$

may be written

$$C_{n,r} = \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\cdots\times\frac{n-r+1}{r}.$$

The numerators of the factors on the right side of this equation begin with n, and form a descending series with the common difference 1; and the denominators begin with 1, and form an ascending series with the common difference 1. Therefore, from some point in the series, these factors become less than 1. Hence, the maximum product is reached when that product includes *all* the factors *greater* than 1.

1. When n is an *odd* number the numerator and the denominator of each factor are alternately both odd and both even, so that the factor greater than 1, but nearest to 1, is the factor whose numerator exceeds the denominator by 2. Hence, in this case, r must have such a value that

$$n - r + 1 = r + 2, \text{ or } r = \frac{n-1}{2}.$$

2. When n is an *even* number the numerator of the first factor is even and the denominator odd; the numerator of the second factor is odd and the denominator even; and so on, alternately, so that the factor greater than 1, but nearest to 1, is the factor whose numerator exceeds the denominator by 1. Hence, in this case, r must have such a value that

$$n - r + 1 = r + 1, \text{ or } r = \frac{n}{2}.$$

(1) What value of r will give the greatest number of combinations out of 7 things?

Here n is odd, and $r = \frac{n-1}{2} = \frac{7-1}{2} = 3.$

$$\therefore C_{7,3} = \frac{7 \times 6 \times 5}{1 \times 2 \times 3} = 35.$$

If $r = 4$, then $C_{7,4} = \frac{7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4} = 35.$

When the number of things is *odd* there are two equal numbers of combinations, namely, when the number of things taken together is *just under* and *just over one-half* of the whole number of things.

(2) What value of r will give the greatest number of selections out of 8 things?

Here n is even, and $r = \frac{n}{2} = \frac{8}{2} = 4.$

$$\therefore C_{8,4} = \frac{8 \times 7 \times 6 \times 5}{1 \times 2 \times 3 \times 4} = 70,$$

so that, when the number of things is *even*, the number of selections will be greatest when *one-half* of the whole are taken together.

346. Division into Two Groups. The number of different ways in which $p + q$ things, all different, can be divided into *two* groups of p things and q things respectively is the same as the number of ways in which p things can be *selected* from $p + q$ things, or $\frac{\lfloor p+q}{\lfloor p \lfloor q}$.

For, to each selection of p things *taken* corresponds a selection of q things *left*, and each selection therefore effects the division into the required groups.

(1) In how many ways can 18 men be divided into 2 groups of 6 and 12 each?

$$\frac{\lfloor 18}{\lfloor 6 \lfloor 12} = 18{,}564 \text{ ways.}$$

(2) A boat's crew consists of 6 men, of whom 2 can row only on the stroke side of the boat, and 1 can row only on the bow side. In how many ways can the crew be arranged?

There are left 3 men who can row on either side; 1 of these must row on the stroke side, and 2 on the bow side.

The number of ways in which these 3 can be selected is

$$\frac{\underline{|3}}{\underline{|2}\,\underline{|1}} = 3 \text{ ways.}$$

When the stroke side is completed the 3 men can be arranged in $\underline{|3}$ ways; likewise, the 3 men of the bow side can be arranged in $\underline{|3}$ ways. Hence, the arrangement can be made in

$$3 \times \underline{|3} \times \underline{|3} = 108 \text{ ways.}$$

347. Division into Three or More Groups. The number of different ways in which $p+q+r$ things, all different, can be divided into *three* groups of p things, q things, and r things respectively is $\dfrac{\underline{|p+q+r}}{\underline{|p}\,\underline{|q}\,\underline{|r}}$.

For, $p+q+r$ things may be divided into *two* groups of p things and $q+r$ things in $\dfrac{\underline{|p+q+r}}{\underline{|p}\,\underline{|q+r}}$ ways; then, the group of $q+r$ things may be divided into *two* groups of q things and r things in $\dfrac{\underline{|q+r}}{\underline{|q}\,\underline{|r}}$ ways. Hence, the division into *three* groups may be effected in

$$\frac{\underline{|p+q+r}}{\underline{|p}\,\underline{|q+r}} \times \frac{\underline{|q+r}}{\underline{|q}\,\underline{|r}} \text{ or } \frac{\underline{|p+q+r}}{\underline{|p}\,\underline{|q}\,\underline{|r}} \text{ ways;}$$

and so on, for any number of groups.

In how many ways can a company of 100 soldiers be divided into 3 squads of 50, 30, and 20 respectively?

The answer is $\dfrac{\underline{|100}}{\underline{|50}\,\underline{|30}\,\underline{|20}}$ ways.

348. When the number of things is the *same* in two or more groups, *and there is no distinction to be made between these groups*, the number of ways given by the preceding section is too large.

(1) Divide the letters *a*, *b*, *c*, *d* into two groups of 2 letters each.

The number of ways given by § 346 is $\dfrac{\lfloor\underline{4}}{\lfloor\underline{2}\,\lfloor\underline{2}} = 6$; these ways are:

I. *ab* *cd*.	III. *ad* *bc*.	V. *bd* *ac*.
II. *ac* *bd*.	IV. *bc* *ad*.	VI. *cd* *ab*.

Since there is no distinction between the groups, IV is the same as III, V the same as II, and VI the same as I.

Hence, the correct answer is $\dfrac{1}{2} \times \dfrac{\lfloor\underline{4}}{\lfloor\underline{2}\,\lfloor\underline{2}}$, or 3.

If, however, a distinction is to be made between the two groups in any one division, the answer is 6.

In the case of three similar groups the result given by § 347 is to be divided by $\lfloor\underline{3}$, the number of ways in which three groups can be arranged among themselves; in the case of four groups, by $\lfloor\underline{4}$; and so on, for any number of groups.

(2) In how many ways can 18 men be divided into two groups of 9 each?

According to § 346, the answer would be $\dfrac{\lfloor\underline{18}}{\lfloor\underline{9}\,\lfloor\underline{9}}$.

The two groups, considered as groups, have no distinction; therefore, permuting them gives no new arrangement, and the true result is obtained by dividing the preceding by $\lfloor\underline{2}$, and is $\dfrac{\lfloor\underline{18}}{\lfloor\underline{2}\,\lfloor\underline{9}\,\lfloor\underline{9}}$.

If any condition is added that will make the two groups *different*, — if, for example, one group wear red badges and the other blue, — then the answer will be $\dfrac{\lfloor\underline{18}}{\lfloor\underline{9}\,\lfloor\underline{9}}$.

(3) In how many ways can a pack of 52 cards be divided equally among 4 players, A, B, C, D?

Here the assignment of a particular group to a *different* player makes the *division* different, and there is, therefore, a distinction between the groups; the answer is $\dfrac{\lfloor\underline{52}}{\lfloor\underline{13}\,\lfloor\underline{13}\,\lfloor\underline{13}\,\lfloor\underline{13}}$.

(4) In how many ways can 52 cards be divided into 4 piles of 13 each?

Here there is no distinction between the groups, and the answer is

$$\frac{\underline{|52}}{\underline{|4}\,\underline{|13}\,\underline{|13}\,\underline{|13}\,\underline{|13}}.$$

Exercise 49

1. How many numbers of 5 figures each can be formed with the digits 1, 2, 3, 4, 5, no digit being repeated?

2. How many *even* numbers of 4 figures each can be formed with the digits 1, 2, 3, 4, 5, 6, no digit being repeated?

3. How many *odd* numbers between 1000 and 5000 can be formed with the figures 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, no figure being repeated? How many of these numbers will be divisible by 5?

4. How many three-lettered words can be made from the alphabet, no letter being repeated in the same word?

5. In how many ways can 4 persons, A, B, C, D, sit at a round table?

6. In how many ways can 6 persons form a ring?

7. How many words can be made with 9 letters, 3 letters remaining inseparable and keeping the same order?

8. What will be the answer to the preceding problem if the 3 inseparable letters can be arranged in any order?

9. A captain, having under his command 60 men, wishes to form a guard of 8 men. In how many different ways can the guard be formed?

10. A detachment of 30 men must furnish each night a guard of 4 men. For how many nights can a different guard be formed, and how many times will each soldier serve?

11. Out of 12 Democrats and 16 Republicans, how many different committees can be formed, each committee consisting of 3 Democrats and 4 Republicans?

12. Out of 26 Republicans and 14 Democrats, how many different committees can be formed, each committee consisting of 10 Republicans and 8 Democrats?

13. There are m different things of one kind and n different things of another kind; how many different sets can be made, each set containing r things of the first kind and s of the second?

14. With 12 consonants and 6 vowels, how many different words can be formed consisting of 3 different consonants and 2 different vowels, any arrangement of letters being considered a word?

15. With 10 consonants and 6 vowels, how many words can be formed, each word containing 5 consonants and 4 vowels?

16. How many words can be formed with 20 consonants and 6 vowels, each word containing 3 consonants and 2 vowels, the vowels occupying the second and fourth places?

17. An assembly of stockholders, composed of 40 merchants, 20 lawyers, and 10 physicians, wishes to elect a commission of 4 merchants, 1 physician, and 2 lawyers. In how many ways can the commission be formed?

18. Of 8 men forming a boat's crew, 1 is selected as stroke. How many arrangements of the rest are possible? When the 4 men who row on each side are decided on, how many arrangements are still possible?

19. A boat's crew consists of 8 men. Either A or B must row stroke. Either B or C must row bow. D can pull only

on the starboard side. In how many ways can the crew be seated?

NOTE. Stroke and bow are on opposite sides of the boat.

20. A boat's crew consists of 8 men. Of these, 3 can row only on the port side, and 2 can row only on the starboard side. In how many ways can the crew be seated?

21. Of a base ball nine, either A or B must pitch; either B or C must catch; D, E, and F must play in the outfield. In how many ways can the nine be arranged?

22. How many signals may be made with 8 flags of different colors, which can be hoisted either singly, or any number at a time, one above another?

23. Of 30 things, how many must be taken together in order that, having that number for selection, there may be the greatest possible variety of choice?

24. The number of combinations of $n + 2$ objects, taken 4 at a time, is to the number of combinations of n objects, taken 2 at a time, as 11 is to 1. Find n.

25. The number of combinations of n things, taken r together, is 3 times the number of combinations when $r - 1$ are taken together, and half the number of combinations when $r + 1$ are taken together. Find n and r.

26. At a game of cards, 3 being dealt to each person, any one can have 425 times as many hands as there are cards in the pack. How many cards are there in the pack?

27. It is proposed to divide 15 objects into lots, each lot containing 3 objects. In how many ways can the lots be made?

28. The number of combinations of $2n$ things, taken $n - 1$ together, is to the number of combinations of $2(n - 1)$ things, taken n together, as 132 to 35. Find n.

349. Permutations, Repetitions allowed. Suppose we have n letters, which are all different, and that *repetitions* are allowed.

Then, in making any arrangement, the first place can be filled in n ways.

When the first place has been filled the second place can be filled in n ways, since repetitions are allowed. Hence, the first two places can be filled in $n \times n$, or n^2, ways (§ 336).

Similarly, the first three places can be filled in $n \times n \times n$, or n^3, ways (§ 337).

In general, r places can be filled in n^r ways; or, *the number of arrangements of* n *different things taken* r *at a time, when repetitions are allowed, is* n^{r}.

(1) How many three-lettered words can be made from the alphabet when repetitions are allowed.

Here the first place can be filled in 26 ways; the second place in 26 ways; and the third place in 26 ways. The number of words is, therefore, $26^3 = 17{,}576$.

(2) In the common system of notation how many numbers can be formed, each number consisting of not more than 5 figures?

Each of the possible numbers may be regarded as consisting of 5 figures, by prefixing zeros to the numbers consisting of less than 5 figures. Thus, 247 may be written 00247.

Hence, every possible arrangement of 5 figures out of the 10 figures, except 00000, will give one of the required numbers, and the answer is $10^5 - 1 = 99{,}999$, that is, all the numbers between 0 and 100,000.

350. Permutations, Things Alike, All together. Consider the number of arrangements of the letters a, a, b, b, b, c, d.

Suppose the a's to be different and the b's to be different, and distinguish them by a_1, a_2, b_1, b_2, b_3.

The 7 letters can now be arranged in $\underline{|7}$ ways (§ 339).

Now suppose the two a's to become alike, and the three b's to become alike. Then, where we before had $\underline{|2}$ arrangements of the a's among

themselves, we now have but one arrangement, *aa*; and where we before had $\underline{|3}$ arrangements of the *b*'s among themselves, we now have but one arrangement, *bbb*.

Hence, the number of arrangements is $\dfrac{\underline{|7}}{\underline{|2}\,\underline{|3}} = 420.$

In general, the number of arrangements of n *things, of which* p *are alike,* q *others are alike, and* r *others are alike,* $\cdots$, *is*

$$\frac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}\cdots}.$$

(1) In how many ways can the letters of the word *college* be arranged?

If the two *l*'s were different and the two *e*'s were different, the number of ways would be $\underline{|7}$. Instead of two arrangements of the two *l*'s, we have but one arrangement, *ll*; and instead of two arrangements of the two *e*'s, we have but one arrangement, *ee*. Hence, the number of ways is

$$\frac{\underline{|7}}{\underline{|2}\,\underline{|2}} = 1260.$$

(2) In how many ways can the letters of the word *Mississippi* be arranged?

$$\frac{\underline{|11}}{\underline{|4}\,\underline{|4}\,\underline{|2}} = 34,650.$$

(3) In how many different orders can a row of 4 white balls and 3 black balls be arranged?

$$\frac{\underline{|7}}{\underline{|4}\,\underline{|3}} = 35.$$

351. Combinations, Repetitions allowed. We shall illustrate by two examples the method of solving problems which come under this head.

(1) In how many ways can a selection of 3 letters be made from the letters *a*, *b*, *c*, *d*, *e*, if repetitions are allowed?

The selections will be of three classes:

(a) All three letters alike.
(b) Two letters alike.
(c) The three letters all different.

(a) There will be 5 selections, since any one of the 5 letters may be taken 3 times.

(b) Any one of the 5 letters may be taken twice, and with these may be put any one of the other 4 letters. Hence, the number of selections is 5×4, or 20.

(c) The number of selections (§ 341) is $\frac{5 \times 4 \times 3}{1 \times 2 \times 3}$, or 10.

Hence, the total number of selections is $5 + 20 + 10 = 35$.

(2) How many different throws can be made with 4 dice?

The throws may be divided into five classes:

(a) All four dice alike.
(b) Three dice alike.
(c) Two dice alike, and the other two alike.
(d) Two dice alike, and the other two different.
(e) The four dice different.

(a) There are 6 throws.

(b) Any of the 6 numbers may be taken 3 times, and with these may be put any one of the other 5 remaining numbers. Hence, the number of throws is 6×5, or 30.

(c) Any two of the 6 pairs of doublets may be selected. Hence, the number of throws is $\frac{6 \times 5}{1 \times 2}$, or 15.

(d) Any pair of doublets may be put with any selection of 2 different numbers from the remaining 5. Hence, the number of throws is

$$6 \times \frac{5 \times 4}{1 \times 2} = 60.$$

(e) The number of throws is $\frac{6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4} = 15$.

The answer is, then, $6 + 30 + 15 + 60 + 15 = 126$.

352. Combinations and Permutations, Things Alike. We shall illustrate by an example the method of solving problems which come under this head.

How many selections of 4 letters each can be made from the letters in the word *proportion?* How many arrangements of 4 letters each?

There are 10 letters as follows:

$$\begin{array}{cccccc} o & p & r & t & i & n \\ o & p & r & & & \\ o & & & & & \end{array}$$

Selections: The selections may be divided into four classes:

(a) Three letters alike.
(b) Two letters alike, two others alike.
(c) Two letters alike, other two different.
(d) Four letters different.

(a) With the 3 o's we may put any one of the 5 other letters, giving 5 selections.

(b) We may choose any 2 out of the 3 pairs, o, o; p, p; r, r.

$$\frac{3 \times 2}{1 \times 2} = 3 \text{ selections.}$$

(c) With any one of the 3 pairs we can put any two of the 5 remaining letters in the first line.

$$3 \times \frac{5 \times 4}{1 \times 2} = 30 \text{ selections.}$$

(d) $$\frac{6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4} = 15 \text{ selections.}$$

Hence, the total number of *selections* is $5 + 3 + 30 + 15 = 53$.

Arrangements: (a) Each selection can be arranged in $\frac{|\underline{4}}{|\underline{3}} = 4$ ways.

$$5 \times 4 = 20 \text{ arrangements.}$$

(b) Each selection can be arranged in $\frac{|\underline{4}}{|\underline{2}\,|\underline{2}} = 6$ ways.

$$3 \times 6 = 18 \text{ arrangements.}$$

(c) Each selection can be arranged in $\frac{|\underline{4}}{|\underline{2}} = 12$ ways.

$$30 \times 12 = 360 \text{ arrangements.}$$

(d) Each selection can be arranged in $|\underline{4} = 24$ ways.

$$15 \times 24 = 360 \text{ arrangements.}$$

Hence, the total number of *arrangements* is

$$20 + 18 + 360 + 360 = 758.$$

353. Total Number of Combinations. I. *The whole number of ways in which a combination (of some, or all) can be made from* n *different things is* $2^n - 1$.

For, each thing can be either taken or left; that is, can be disposed of in 2 ways.

There are n things; hence (§ 337), they can all be disposed of in 2^n ways. But among these ways is included the case in which all are rejected; and this case is inadmissible.

Hence, the number of ways of making a selection is $2^n - 1$.

(1) In a shop window 20 different articles are exposed for sale. What choice has a purchaser?

The number of ways in which a purchaser may make a selection is

$$2^{20} - 1 = 1,048,575.$$

(2) How many different amounts can be weighed with 1-pound, 2-pound, 4-pound, 8-pound, and 16-pound weights?

The number of different amounts that can be weighed is

$$2^5 - 1 = 31.$$

NOTE. Let the student write out the 31 weights.

II. *The whole number of ways in which a selection can be made from* $p + q + r + \cdots$ *things, of which* p *are alike,* q *are alike,* r *are alike,* $\cdots$, *is* $\{(p+1)(q+1)(r+1)\cdots\} - 1$.

For, the set of p things may be disposed of in $p + 1$ ways, since none of them may be taken, or 1, 2, 3, $\cdots$, or p, may be taken.

In like manner, the q things may be disposed of in $q + 1$ ways; the r things in $r + 1$ ways; and so on.

Hence (§ 337), all the things may be disposed of in

$$(p+1)(q+1)(r+1)\cdots \text{ ways.}$$

But the case in which *all* the things are rejected is inadmissible; hence, the whole number of ways is

$$\{(p+1)(q+1)(r+1)\cdots\} - 1.$$

In how many ways can 2 boys divide between them 10 oranges all alike, 15 apples all alike, and 20 peaches all alike?

Here the case in which the first boy takes none, and the case in which the second boy takes none, must be rejected.

Therefore, the answer is one less than the result, according to II.

$$11 \times 16 \times 21 - 2 = 3694.$$

Exercise 50

1. How many three-lettered words can be made from the 6 vowels when repetitions are allowed?

2. A railway signal has 3 arms, and each arm may take 4 different positions, including the position of rest. How many signals in all can be made?

3. In how many different orders can a row of 7 white balls, 2 red balls, and 3 black balls be arranged?

4. In how many ways can the letters of the word *mathematics*, taken all together, be arranged?

5. How many different signals can be made with 10 flags, of which 3 are white, 2 red, and the rest blue, always hoisted all together and one above another?

6. How many signals can be made with 7 flags, of which 2 are red, 1 white, 3 blue, and 1 yellow, always displayed all together and one above another?

7. In how many ways can 5 letters be selected from a, b, c, d, e, f, if each letter may be taken once, twice, three times, four times, or five times, in making the selection?

8. In how many ways can 6 rugs be selected at a shop where two kinds of rugs are sold?

9. How many dominos are there in a set numbered from double blank to double ten?

10. In how many ways can 3 letters be selected from n different letters, when repetitions are allowed?

11. Five flags of different colors can be hoisted either singly, or any number at a time, one above another. How many different signals can be made with them?

12. If there are m kinds of things, and 1 thing of the first kind, 2 of the second, 3 of the third, and so on, in how many ways can a selection be made?

13. How many selections of 6 letters each can be made from the letters in the word *democracy?* How many arrangements of 6 letters each?

14. If of $p + q + r$ things, p are alike, and q are alike, and the rest different, show that the total number of selections is $(p + 1)(q + 1)2^r - 1$.

15. Show that the total number of arrangements of $2n$ letters, of which some are a's and the rest b's, is greatest when the number of a's is equal to the number of b's.

16. If in a given number the prime factor a occurs m times, the prime factor b, n times, the prime factor c, p times, and these are all the factors, find the number of different divisors of the given number.

17. If P_n represents the total number of permutations of n different letters, $a_1, a_2, a_3, \cdots, a_n$, and Q_n represents the number of arrangements in which no letter occupies the place denoted by its index (the *complete disarrangement*), show that

$$Q_2 = P_2 - 2P_1 + P_0, \qquad P_0 = 1,$$
$$Q_3 = P_3 - 3P_2 + 3P_1 - P_0,$$
$$Q_4 = P_4 - 4P_3 + 6P_2 - 4P_1 + P_0,$$

and, in general,
$$Q_n = P_n - \frac{n}{1}P_{n-1} + \frac{n(n-1)}{1 \times 2}P_{n-2} - \frac{n(n-1)(n-2)}{1 \times 2 \times 3}P_{n-3} + \cdots$$

CHAPTER XXIII

CHANCE

354. Definitions. If an event can happen in a ways and fail in b ways, and all these $a + b$ ways are *equally likely* to occur; if, also, one, *and only one*, of these $a + b$ ways *can* occur, and one *must* occur; then, the **chance** of the event *happening* is $\frac{a}{a+b}$, and the chance of the event *failing* is $\frac{b}{a+b}$.

Thus, let the event be the throwing of an even number with a single die.

The event can happen in 3 ways, by the die turning up a two, a four, or a six; and fail in 3 ways, by the die turning up a one, a three, or a five; and all these 6 ways are equally likely to occur.

Moreover, one, and only one, of these 6 ways *can* occur, and one *must* occur (for it is assumed that the die is to be thrown).

Consequently, by the definition, the chance of throwing an even number is $\frac{3}{3+3}$, or $\frac{1}{2}$; and the chance of throwing a number not even, that is, odd, is $\frac{3}{3+3}$, or $\frac{1}{2}$.

The above may be regarded as giving a definition of the term *chance* as that term is used in mathematical works. Instead of *chance*, *probability* is often used.

355. Odds. In the case of the event in § 354 the *odds* are said to be a to b *in favor of* the event, if a is greater than b; and b to a *against* the event, if b is greater than a.

If $a = b$, the odds are said to be *even* on the event.

Thus, the odds are 5 to 1 against throwing a six in one throw with a single die, since there are 5 unfavorable ways and 1 favorable way, and all these 6 ways are equally likely to occur.

356. Rules. From the definitions it is evident that,

The chance of an event happening is expressed by the fraction of which the numerator is the number of favorable ways, and the denominator the whole number of ways favorable and unfavorable.

For example, take the throwing of a six with a single die. The number of favorable ways is 1; the whole number of ways is 6. Hence, the chance of throwing a six with a single die is $\frac{1}{6}$.

The chance of an event not happening is expressed by the fraction of which the numerator is the number of unfavorable ways, and the denominator the whole number of ways favorable and unfavorable.

For example, take the throwing of a six with a single die. The number of unfavorable ways is 5; the whole number of ways is 6. Hence, the chance of not throwing a six with a single die is $\frac{5}{6}$.

357. Certainty. If the event is *certain* to happen, there are no ways of failing, and $b = 0$. The chance of the event happening is then $\frac{a}{a+0} = 1$. Hence, *certainty* is expressed by 1.

It is to be observed that the fraction which expresses a *chance* or *probability* is less than 1, unless the event is certain to happen, in which case the chance of the event happening is 1.

358. Since
$$\frac{a}{a+b} + \frac{b}{a+b} = 1,$$
we have
$$\frac{b}{a+b} = 1 - \frac{a}{a+b}.$$

Hence, if p is the chance of an event happening, $1 - p$ is the chance of the event failing.

359. Examples; Simple Event. (1) What is the chance of throwing double sixes in one throw with 2 dice?

Each die may fall in 6 ways, and all these ways are equally likely to occur. Hence, the 2 dice may fall in 6×6, or 36, ways (§ 336), and

these 36 ways are all equally likely to occur. Moreover, only *one* of the 36 ways *can* occur, and one *must* occur.

There is only one way which will give double sixes. Hence, the chance of throwing double sixes is $\frac{1}{36}$.

REMARK. It may seem as though the number of ways in which the dice can fall ought to be 21, the number of different throws that can be made with two dice. These throws, however, are not all *equally likely* to occur.

To obtain ways that are equally likely to occur we must go back to the case of a single die. One die can fall in 6 ways, and from the *construction of the die* it is evident that these 6 ways are all equally likely to occur.

Also the second die can fall in 6 ways, all equally likely to occur. Hence, the 2 dice can fall in 36 ways, all equally likely to occur (§ 336).

In this case the throw, first die five, second die six, is considered a different throw from first die six, second die five. Consequently, the chance of throwing a five and a six is $\frac{2}{36}$, or $\frac{1}{18}$, while the chance of throwing double sixes is only $\frac{1}{36}$. This verifies the statement already made, that the 21 different throws are not all equally likely to occur.

(2) What is the chance of throwing one, and only one, five in one throw with two dice?

The whole number of ways, all equally likely to occur, in which the dice can fall is 36. In 5 of these 36 ways the first die will be a five, and the second die not a five; in five of these 36 ways the second die will be a five, and the first not a five. Hence, in 10 of these ways one die, and only one die, will be a five; and the required chance is $\frac{10}{36}$, or $\frac{5}{18}$.

The odds are 13 to 5 against the event.

(3) In the same problem what is the chance of throwing *at least* one five?

Here we have to include also the way in which both dice fall fives, and the required chance is $\frac{11}{36}$.

The odds are 25 to 11 against the event.

(4) What is the chance of throwing a total of 5 in one throw with 2 dice?

The whole number of ways, all equally likely to occur, in which the dice can fall is 36. Of these ways 4 give a total of 5; *viz.*, 1 and 4, 2 and 3, 3 and 2, 4 and 1. Hence, the required chance is $\frac{4}{36}$, or $\frac{1}{9}$.

The odds are 8 to 1 against the event.

(5) From an urn containing 5 black and 4 white balls, 3 balls are to be drawn at random. Find the chance that 2 balls will be black and 1 white.

There are 9 balls in the urn. The whole number of ways in which 3 balls can be selected from 9 is $\frac{9 \times 8 \times 7}{1 \times 2 \times 3}$, or 84.

From the 5 black balls 2 can be selected in $\frac{5 \times 4}{1 \times 2}$, or 10, ways; from the 4 white balls 1 can be selected in 4 ways; hence, 2 black balls and 1 white ball can be selected in 10×4, or 40, ways.

The required chance is $\frac{40}{84} = \frac{10}{21}$.

The odds are 11 to 10 against the event.

(6) From a bag containing 10 balls 4 are drawn and replaced; then 6 are drawn. Find the chance that the 4 first drawn are among the 6 last drawn.

The second drawing could be made altogether in

$$\frac{\lfloor 10}{\lfloor 6 \lfloor 4} = 210 \text{ ways.}$$

But the drawing can be made so as to include the 4 first drawn in

$$\frac{\lfloor 6}{\lfloor 2 \lfloor 4} = 15 \text{ ways,}$$

since the only choice consists in selecting 2 balls from the 6 not previously drawn. Hence, the required chance is $\frac{15}{210} = \frac{1}{14}$.

(7) If 4 coppers are tossed, what is the chance that exactly 2 will turn up heads?

Since each coin may fall in 2 ways, the 4 coins may fall in $2^4 = 16$ ways (§ 337). The 2 coins to turn up heads can be selected from the 4 coins in $\frac{4 \times 3}{1 \times 2} = 6$ ways. Hence, the required chance is $\frac{6}{16} = \frac{3}{8}$.

The odds are 5 to 3 against the event.

(8) In one throw with 2 dice, which sum is more likely to be thrown, 9 or 12?

Out of the 36 possible ways of falling, *four* give the sum 9 (namely, $6 + 3$, $3 + 6$, $5 + 4$, $4 + 5$), and *only one* way gives 12 (namely, $6 + 6$). Hence, the chance of throwing 9 is *four times* that of throwing 12.

Note. It will be observed in the above examples that we sometimes use arrangements and sometimes use selections. In some problems the former, in some problems the latter, will give the ways which are all *equally likely* to occur.

In some problems we can use either selections or arrangements.

Exercise 51

1. The chance of an event happening is $\frac{4}{7}$. What are the odds in favor of the event?

2. If the odds are 10 to 1 against an event, what is the chance of the event happening?

3. The odds against an event are 3 to 1. What is the chance of the event happening?

4. The chance of an event happening is $\frac{2}{9}$. Find the odds against the event.

5. In one throw with a pair of dice what number is most likely to be thrown? Find the odds against throwing that number.

6. Find the chance of throwing doublets in one throw with a pair of dice.

7. If 4 cards are drawn from a pack of 52 cards, what is the chance that there will be 1 of each suit?

8. If 4 cards are drawn from a pack of 52 cards, what is the chance that they will all be hearts?

9. If 10 persons stand in a line, what is the chance that 2 assigned persons will stand together?

10. If 10 persons form a ring, what is the chance that 2 assigned persons will stand together?

11. Three balls are to be drawn from an urn containing 5 black, 3 red, and 2 white balls. What is the chance of drawing 1 red ball and 2 black balls?

12. In a bag are 5 white and 4 black balls. If 4 balls are drawn, what is the chance that they will all be of the same color?

13. If 2 tickets are drawn from a package of 20 tickets marked 1, 2, 3, $\cdots$, what is the chance that both will be marked with *odd* numbers?

14. A bag contains 3 white, 4 black, and 5 red balls; 3 balls are drawn. Find the odds against the 3 being of three different colors.

15. Show that the odds are 35 to 1 against throwing 16 in a single throw with 3 dice.

16. There are 10 tickets numbered 1, 2, $\cdots$, 9, 0. Three tickets are drawn at random. Find the chance of drawing a total of 22.

17. Find the probability of throwing 15 in one throw with 3 dice.

18. With 3 dice, what are the relative chances of throwing a doublet and a triplet?

19. If 3 cards are drawn from a pack of 52 cards, what is the chance that they will be king, queen, and knave?

360. Dependent and Independent Events. Thus far we have considered only single events. We proceed to cases in which there are two or more events.

Two or more events are **dependent** or **independent**, according as the happening (or failing) of one event *does* or *does not* affect the happening (or failing) of the other events.

Thus, throwing a six and throwing a five in any particular throw with one die are *dependent* events, since the happening of one *excludes* the happening of the other.

But, with 2 dice, throwing a six with one die and throwing a five with the other are *independent* events, since the happening of one has no effect upon the happening of the other.

361. Events mutually Exclusive. When several dependent events are so related that one, and *only one*, of the events can happen, the events are said to be mutually exclusive.

Thus, let a single die be thrown, and regard its falling one up, two up, three up, and so on, as six different events. Then, these six events are evidently mutually exclusive.

362. *If there are several events of which one, and only one, can happen, the chance that one will happen is the* **sum** *of the respective chances of happening.*

To prove this, let a, a', a'', $\cdots$ be the number of ways favorable to the first, second, third, $\cdots$ events respectively, and m the number of ways unfavorable to *all* the events, these $a + a' + a'' + \cdots + m$ ways being all equally likely to occur, and such that one *must* occur.

Represent by n the sum $a + a' + a'' + \cdots + m$.

Of the n ways which may occur, a, a', a'', $\cdots$ ways are favorable to the first, second, third, $\cdots$ events respectively.

Hence, the respective chances of happening are

$$\frac{a}{n},\ \frac{a'}{n},\ \frac{a''}{n},\ \cdots$$

Of the n ways which may occur, $a + a' + a'' + \cdots$ ways are favorable to the happening of *some one* of the events. Hence, the chance that some one of the events will happen is

$$\frac{a + a' + a'' + \cdots}{n}, \text{ or } \frac{a}{n} + \frac{a'}{n} + \frac{a''}{n} + \cdots$$

If, then, p, p', p'', $\cdots$ are the respective chances of happening of the first, second, third, $\cdots$ of several mutually exclusive events, the chance that *some one* of the events will happen is $p + p' + p'' + \cdots$

Thus, let the throwing of two, a four, and a six, with a single die, be three events. These three events are evidently mutually exclusive.

There are 6 ways, all equally likely to occur, in which the die can fall; of these 6 ways, one must occur, and only one can occur.

The chance of throwing a two is $\frac{1}{6}$; of throwing a four, $\frac{1}{6}$; of throwing a six, $\frac{1}{6}$; since there is but one favorable way in each case.

The chance of throwing an even number is $\frac{3}{6}$, since 3 out of the 6 ways are favorable ways.

But $\frac{3}{6} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$; hence, $\frac{3}{6}$ is the sum of the respective chances of throwing a two, a four, a six. (Compare § 354.)

363. Compound Events. If there are two or more events, the happening of them together, or in succession, may be regarded as a compound event.

Thus, the throwing of double sixes with a pair of dice may be regarded as a compound event compounded of the throwing of a six with the first die and the throwing of a six with the second die.

364. Concurring Independent Events. *The chance that two or more independent events will happen together is the* **product** *of the respective chances of happening.*

To prove this, let a and a' be the number of ways favorable to the first and second events respectively, and b and b' the number of ways unfavorable to the first and second events respectively; the $a + b$ ways being all equally likely to occur, and such that one must occur, and only one can occur; and the $a' + b'$ ways being all equally likely to occur, and such that one must occur, and only one can occur.

Then, the respective chances of happenings are $\frac{a}{a+b}$ and $\frac{a'}{a'+b'}$; and the respective chances of failing are $\frac{b}{a+b}$ and $\frac{b'}{a'+b'}$. Represent the former by p and p'; then, the latter will be $1 - p$ and $1 - p'$.

Consider the compound event. It is evident, by § 336, that there are $(a + b)(a' + b')$ ways, all equally likely to occur. Of these, one *must* occur, and only one *can* occur.

The number of ways in which both events can happen is aa'; hence, the chance that both events will happen is

$$\frac{aa'}{(a+b)(a'+b')}=\frac{a}{a+b}\times\frac{a'}{a'+b'}=pp'.$$

Similarly, the chance that both events will fail is

$$\frac{bb'}{(a+b)(a'+b')}=(1-p)(1-p');$$

the chance that the first will happen and the second fail is

$$\frac{ab'}{(a+b)(a'+b')}=p(1-p');$$

the chance that the first will fail and the second happen is

$$\frac{ba'}{(a+b)(a'+b')}=(1-p)p'.$$

Similarly for three or more events.

365. Successive Dependent Events. By a slight change in the meaning of the symbols of § 364, we can find the chance of the happening together of two or more *dependent* events.

For, suppose that, *after the first event has happened*, the second event can follow in a' ways and not follow in b' ways. Then the two events can happen in $\frac{aa'}{(a+b)(a'+b')}$ ways; and so on, as in § 364.

Hence, if p is the chance that the first event will happen, and p' the chance that after the first event has happened the second will follow, pp' is the chance of both happening; $(1-p)(1-p')$, the chance of both failing; and so on.

Similarly for three or more events.

366. Examples. (1) What is the chance of throwing double sixes in one throw with 2 dice?

Regard this as a *compound* event. The chance that the first die will turn up a six is $\frac{1}{6}$; the chance that the second die will turn up a six is $\frac{1}{6}$; the chance that each die will turn up a six is $\frac{1}{6}\times\frac{1}{6}$, or $\frac{1}{36}$.

The events are here *independent*. In Example (1), § 359, the throwing of double sixes is regarded as a *simple* event.

(2) What is the chance of throwing one, and only one, five in a single throw with 2 dice?

The chance that the first die will be a five, and the second not a five, is $\frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$; the chance that the first die will not be a five, and the second die a five, is $\frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$. These two events are dependent and mutually exclusive, and the chance that one or the other of them will happen is (§ 362) $\frac{5}{36} + \frac{5}{36} = \frac{5}{18}$. (Compare Example (2), § 359.)

(3) What is the chance of throwing a total of 5 in one throw with 2 dice?

There are four ways of throwing 5: 1 and 4, 2 and 3, 3 and 2, 4 and 1. The chance of each of these ways happening is $\frac{1}{36}$. The events are mutually exclusive; hence, the chance of some one happening is (§ 362) $\frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{1}{9}$. (Compare Example (4), § 359.)

(4) A bag contains 3 balls, 2 of which are white; another bag contains 6 balls, 5 of which are white. If a person is to draw 1 ball from each bag, what is the chance that both balls drawn will be white?

The chance that the ball drawn from the first bag will be white is $\frac{2}{3}$; the chance that the ball drawn from the second bag will be white is $\frac{5}{6}$. The events are independent; hence, the chance that both balls will be white is $\frac{2}{3} \times \frac{5}{6} = \frac{5}{9}$ (§ 364).

(5) In the last example, if all the balls are in one bag, and 2 balls are to be drawn, what is the chance that both balls will be white?

The chance that the first ball will be white is $\frac{7}{9}$; the chance that, after 1 white ball has been drawn, the second will be white is $\frac{6}{8}$; the chance of drawing 2 white balls is (§ 365) $\frac{7}{9} \times \frac{6}{8} = \frac{7}{12}$.

(6) The chance that A can solve this problem is $\frac{3}{4}$; the chance that B can solve it is $\frac{5}{12}$. If both try, what is the chance (1) that both solve it; (2) that A solves it, and B fails; (3) that A fails, and B solves it; (4) that both fail?

A's chance of success is $\frac{3}{4}$; A's chance of failure is $\frac{1}{4}$.
B's chance of success is $\frac{5}{12}$; B's chance of failure is $\frac{7}{12}$.

Therefore, the chance of (1) is $\frac{2}{3} \times \frac{5}{12} = \frac{10}{36}$;
the chance of (2) is $\frac{2}{3} \times \frac{7}{12} = \frac{14}{36}$;
the chance of (3) is $\frac{1}{3} \times \frac{5}{12} = \frac{5}{36}$;
the chance of (4) is $\frac{1}{3} \times \frac{7}{12} = \frac{7}{36}$.

The sum of these four chances is $\frac{10}{36} + \frac{14}{36} + \frac{5}{36} + \frac{7}{36} = 1$, as it ought to be, since one of the four results is *certain* to happen.

(7) In Example (6) what is the chance that the problem will be solved?

The chance that *both fail* is $\frac{7}{36}$. Hence, the chance that *both do not fail*, or that the problem will be solved, is $1 - \frac{7}{36} = \frac{29}{36}$ (§ 358).

(8) From an urn containing 5 black and 4 white balls, 3 balls are to be drawn at random. Find the chance that of the 3 balls drawn 2 will be black and 1 white.

There are 9 balls in the urn. Suppose the balls to be drawn one at a time. The white ball may be either the first, second, or third ball drawn. In other words, 1 white ball and 2 black balls may be drawn in

$$\frac{\underline{|3}}{\underline{|1}\,\underline{|2}} = 3 \text{ ways (§ 346).}$$

The chance of the order, white black black, is $\frac{4}{9} \times \frac{5}{8} \times \frac{4}{7} = \frac{10}{63}$.
The chance of the order, black white black, is $\frac{5}{9} \times \frac{4}{8} \times \frac{4}{7} = \frac{10}{63}$.
The chance of the order, black black white, is $\frac{5}{9} \times \frac{4}{8} \times \frac{4}{7} = \frac{10}{63}$.
Hence, the required chance is $\frac{10}{63} + \frac{10}{63} + \frac{10}{63} = \frac{10}{21}$ (§ 362).

The method of Example (5), § 359, is, however, recommended for problems of this nature.

(9) When 6 coins are tossed what is the chance that *one*, and *only one*, will fall with the head up?

The chance that the first alone falls with the head up is (§ 364) $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{64}$; the chance that the second alone falls with the head up is $\frac{1}{64}$; and so on, for each of the 6 coins.

Hence, the chance that some one coin, and only one coin, falls with the head up is $\frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} = \frac{6}{64} = \frac{3}{32}$.

(10) When 6 coins are tossed what is the chance that *at least* one will fall with the head up?

The chance that *all* will fall heads down is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{64}$. Hence, the chance that this will not happen is $1 - \frac{1}{64} = \frac{63}{64}$.

(11) A purse contains 9 silver dollars and 1 gold eagle, and another contains 10 silver dollars. If 9 coins are taken out of the first purse and put into the second, and then 9 coins are taken out of the second and put into the first purse, which purse now is the more likely to contain the gold coin?

The gold eagle will not be in the second purse unless it (1) was among the 9 coins taken out of the first and put into the second purse; (2) and *not* among the 9 coins taken out of the second and put into the first purse. The chance of (1) is $\frac{9}{10}$, and when (1) has happened the chance of (2) is $\frac{10}{19}$. Hence, the chance of *both* happening is $\frac{9}{10} \times \frac{10}{19} = \frac{9}{19}$. Therefore, the chance that the eagle is in the second purse is $\frac{9}{19}$, and the chance that it is in the first purse is $1 - \frac{9}{19} = \frac{10}{19}$. Since $\frac{10}{19}$ is greater than $\frac{9}{19}$, the gold coin is more likely to be in the first purse than in the second.

NOTE. The *expectation* from an uncertain event is the product of *the chance* that the event will happen by *the amount* to be realized in case the event happens.

(12) In a bag are 2 red and 3 white balls. A is to draw a ball, then B, and so on alternately; and whichever draws a white ball first is to receive \$10. Find their expectations.

A's chance of drawing a *white* ball at the first trial is $\frac{3}{5}$. B's chance of *having a trial* is equal to A's chance of drawing a *red* ball, or $\frac{2}{5}$. In case A drew a red ball, there would be 1 red and 3 white balls left in the bag, and B's chance of drawing a white ball would be $\frac{3}{4}$. Hence, B's chance of having the trial and drawing a white ball is $\frac{3}{4} \times \frac{2}{5} = \frac{3}{10}$; and B's chance of drawing a red ball is $\frac{1}{4} \times \frac{2}{5} = \frac{1}{10}$.

A's chance of *having a second trial* is equal to B's chance of drawing a *red* ball, or $\frac{1}{10}$. In case B drew a red ball, there would be 3 white balls left, and A's chance of drawing a white ball would be *certainty*, or 1.

A's chance, therefore, is $\frac{3}{5} + \frac{1}{10} = \frac{7}{10}$; and B's chance is $\frac{3}{10}$.

A's expectation, then, is \$7, and B's \$3.

367. Repeated Trials. Given the chance of an event happening in one trial, to find the chance of its happening exactly once, twice, $\cdots$, r times in n trials.

Let p be the chance of the event happening, and q the chance of the event failing, in one trial; so that $q = 1 - p$.

In n trials the event may happen exactly n times, $n-1$ times, $n-2$ times, $\cdots$ down to no times. The respective chances of happening are as follows:

n *times*. The required chance, by § 364, is p^n.

n $-$ 1 *times*. The one failure may occur in any one of the n trials; that is, in n ways. The chance of any particular way occurring is $p^{n-1}q$; the required chance is, therefore, $np^{n-1}q$.

n $-$ 2 *times*. The two failures may occur in any two of the n trials; that is, in $\frac{n(n-1)}{\underline{\lvert 2}}$ ways. The chance of any particular way occurring is $p^{n-2}q^2$; the required chance is, therefore, $\frac{n(n-1)}{\underline{\lvert 2}}p^{n-2}q^2$.

r *times*. The $n-r$ failures may occur in any $n-r$ of the n trials; that is, in $\frac{\underline{\lvert n}}{\underline{\lvert n-r}\,\underline{\lvert r}}$ ways. The chance of any particular way occurring is p^rq^{n-r}; the required chance is, therefore, $\frac{\underline{\lvert n}}{\underline{\lvert n-r}\,\underline{\lvert r}}p^rq^{n-r}$.

Similarly, the chance of exactly r failures is $\frac{\underline{\lvert n}}{\underline{\lvert r}\,\underline{\lvert n-r}}p^{n-r}q^r$. The coefficients of the chance of r successes and of r failures are the same, by § 343.

If, then, $(p+q)^n$ is expanded by the binomial theorem, it is evident that the successive terms are the chances that the event will happen exactly n times, $n-1$ times, $\cdots$ down to no times.

The chances that the event will happen *at least* r times in n trials is evidently $p^n + np^{n-1}q + \cdots + \frac{\underline{\lvert n}}{\underline{\lvert n-r}\,\underline{\lvert r}}p^rq^{n-r}$.

NOTE. Since $p+q=1$, we have, whatever the value of n,

$$1 = p^n + np^{n-1}q + \cdots + npq^{n-1} + q^n,$$

a somewhat remarkable equation inasmuch as there exists but one relation between p and q, *viz.*, $p+q=1$.

368. Examples. (1) What is the chance of throwing with a single die a six exactly 3 times in 5 trials? at least 3 times?

There are to be 2 failures. The 2 failures may occur in any 2 of the 5 trials; that is, in $\frac{5 \times 4}{2}$, or 10, ways. In any particular way there will be 3 sixes and 2 failures, and the chance of this way occurring is $(\frac{1}{6})^3(\frac{5}{6})^2$; the chance of throwing *exactly* 3 sixes is, therefore,

$$10\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 = \frac{125}{3888}.$$

The chance of throwing *at least* 3 sixes is found by adding the respective chances of throwing 5 sixes, 4 sixes, 3 sixes; and is

$$(\tfrac{1}{6})^5 + 5(\tfrac{1}{6})^4(\tfrac{5}{6}) + 10(\tfrac{1}{6})^3(\tfrac{5}{6})^2 = \tfrac{23}{648}.$$

(2) A's skill at a game, which cannot be drawn, is to B's skill as 3 to 4. If they play 3 games, what is the chance that A will win more games than B?

Their respective chances of winning a particular game are $\frac{3}{7}$ and $\frac{4}{7}$. For A to win more games than B, he must win all 3 games or 2 games. The chance that A wins all 3 games is $(\frac{3}{7})^3 = \frac{27}{343}$. The chance that A wins any particular set of 2 games out of the 3 games, and that B wins the third game, is $(\frac{3}{7})^2 \times (\frac{4}{7})$. As there are 3 ways of selecting a set of 2 games out of 3, the chance that A wins 2 games, and B wins 1 game, is $3 \times (\frac{3}{7})^2 \times \frac{4}{7} = \frac{108}{343}$. Hence, the chance that A wins more than B is

$$\tfrac{27}{343} + \tfrac{108}{343} = \tfrac{135}{343}.$$

(3) In the last example find B's chance of winning more games than A.

B's chance of winning all 3 games is $(\frac{4}{7})^3 = \frac{64}{343}$. The chance that B wins 2 games, and A wins 1 game, is $3 \times (\frac{4}{7})^2 \times \frac{3}{7} = \frac{144}{343}$. Hence, B's chance of winning more games than A is $\frac{64}{343} + \frac{144}{343} = \frac{208}{343}$.

Notice that A's chance added to B's chance, $\frac{135}{343} + \frac{208}{343}$, is 1. Why should this be so?

(4) A and B throw with a single die alternately, A throwing first; and the one who throws an ace first is to receive a prize of \$110. What are their respective expectations?

The chance of winning the prize at the first throw is $\frac{1}{6}$; of winning at the second throw, $\frac{5}{6} \times \frac{1}{6}$; of winning at the third throw, $(\frac{5}{6})^2 \times \frac{1}{6}$; of winning at the fourth throw, $(\frac{5}{6})^3 \times \frac{1}{6}$; and so on.

Hence, A's chance is $\frac{1}{6} + (\frac{5}{6})^2\frac{1}{6} + (\frac{5}{6})^4\frac{1}{6} + \cdots$, and B's chance is $(\frac{5}{6})\frac{1}{6} + (\frac{5}{6})^3\frac{1}{6} + (\frac{5}{6})^5\frac{1}{6} + \cdots$ Evidently B's chance is $\frac{5}{6}$ of A's chance. Since A's chance + B's chance = 1, A's chance is $\frac{6}{11}$ and B's $\frac{5}{11}$. A's expectation is $\frac{6}{11}$ of \$110, or \$60; and B's $\frac{5}{11}$ of \$110, or \$50.

Exercise 52

1. One of two events must happen. If the chance of the first is $\frac{2}{3}$ that of the other, find the odds against the first.

2. There are three events, A, B, C, of which one must happen, and only one can happen. The odds are 3 to 8 on A, and 2 to 5 on B. Find the odds on C.

3. In one bag are 9 balls and in another 6; and in each bag the balls are marked 1, 2, 3, and so on. If one ball is drawn from each bag, what is the chance that the two balls will have the same number?

4. What is the chance of throwing at least one ace in 2 throws with one die?

5. Find the probability of throwing a number greater than 9 in a single throw with a pair of dice.

6. The chance that A can solve a certain problem is $\frac{1}{4}$, and the chance that B can solve it is $\frac{2}{3}$. What is the chance that the problem will be solved if both try?

7. A, B, C have equal claims for a prize. A says to B: "You and C draw lots, and the winner shall draw lots with me for the prize." Is this fair?

8. A bag contains 5 tickets numbered 1, 2, 3, 4, 5. Three tickets are drawn at random, the tickets not being replaced after drawing. Find the chance of drawing a total of 10.

9. A bag contains 10 tickets, 5 marked 1, 2, 3, 4, 5, and 5 blank. Three tickets are drawn at random, each being replaced before the next is drawn. Find the probability of drawing a total of 10.

10. Find the probability of drawing in Example 9 a total of 10 when the tickets are not replaced.

11. A bag contains four \$10 gold pieces and six silver dollars. A person is entitled to draw 2 coins at random. Find the value of his expectation.

12. Six \$5 gold pieces, four \$3 gold pieces, and 5 coins which are either all gold dollars or all silver dimes are thrown together into a bag. Assuming that the unknown coins are equally likely to be dimes or dollars, what is a fair price to pay for the privilege of drawing at random a single coin?

13. A bag contains six \$5 gold pieces, and 4 other coins which all have the same value. The expectation of drawing at random 2 coins is worth \$8.40. Find the value of each of the unknown coins.

14. Find the probability of throwing at least one ace in 4 throws with a single die.

15. A copper is tossed 3 times. Find the chance that it will fall heads once and tails twice.

16. What is the chance of throwing double sixes at least once in 3 throws with a pair of dice?

17. Two bags contain each 4 black and 3 white balls. A ball is drawn at random from the first bag, and if it is white, it is put into the second bag, and a ball drawn at random from that bag. Find the odds against drawing two white balls.

18. A and B play at chess, and A wins on an average 2 games out of 3. Find the chance of A's winning exactly 4 games out of the first 6, drawn games being disregarded.

19. At tennis A on an average beats B 2 games out of 3. If they play one set, find the chance that A will win by the score of 6 to 2.

20. A and B, two players of equal skill, are playing tennis. A needs 2 games to win the set, and B needs 3 games. Find the chance that A will win the set.

21. If n coins are tossed up, what is the chance that one, and only one, will turn up head?

22. A bag contains n balls. A person takes out one ball, and then replaces it. He does this n times. What is the chance that he has had in his hand every ball in the bag?

23. If on an average 9 ships out of 10 return safely to port, what is the chance that out of 5 ships expected at least 3 will safely return?

24. At tennis A beats B on an average 2 games out of 3; if the score is 4 games to 3 in B's favor, find the chance of A's winning 6 games before B does.

25. The odds against a certain event are 5 to 4, and the odds for another independent event are 6 to 5. Find the chance that at least one of the events will happen.

26. A draws 5 times (replacing) from a bag containing 3 white and 7 black balls, drawing each time one ball; every time he draws a white ball he is to receive $1, and every time he draws a black ball he is to pay 50 cents. What is his expectation?

27. From a bag containing 2 eagles, 3 dollars, and 3 quarter-dollars, A is to draw 1 coin and then B 3 coins; and A, B, and C are to divide equally the value of the remainder. What are their expectations?

28. What is the chance of throwing with a single die a five at least twice in four throws?

369. Existence of Causes. In the problems thus far considered we have been concerned only with future events; we now proceed to a different class of problems, problems of which the following is the general type.

An event has happened. There are several possible causes, of which one *must* have existed, and only one *can* have existed. From the several possible causes a particular cause is selected; required the chance that this was the true cause.

Before proceeding to the general problem we shall consider some examples.

(1) Ten has been thrown with 2 dice. Required the chance that the throw was double fives.

Ten can be thrown in 3 ways: 6, 4; 4, 6; 5, 5. One of these three ways must have occurred, and only one can have occurred.

Before the event the chances that these respective ways *would* occur were all equal.

We shall *assume* that *after the event* the chances that these respective ways *have* occurred are all equal.

Then, precisely as in § 354, the chance that the throw was double fives is $\frac{1}{3}$, and the chance that the throw was a six and a four is $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$.

(2) Fifteen has been thrown with 3 dice. Required the chance that the throw was 3 fives.

Fifteen can be thrown in 10 ways:

6 5 4	5 4 6	4 5 6	6 6 3	3 6 6
6 4 5	5 6 4	4 6 5	6 3 6	5 5 5

One of these 10 ways must have occurred, and only one can have occurred.

Before the event the chances that these respective ways *would* occur were all equal.

We shall *assume* that *after the event* the chances that the respective ways *have* occurred are all equal.

Then, precisely as in § 354, the chance that the throw was 3 fives is $\frac{1}{10}$.

(3) A box contains 4 white balls and 2 black balls. Two balls are drawn at random and put into a second box. From

the second box 1 ball is then drawn and found to be white. Required the chance that the two balls in the second box are both white.

Before the event there were three cases which might exist. These cases, with the respective chances of existence, were as follows:

The second box might contain:

(a) 2 white balls, of which the chance was $\frac{2}{5}$.

(b) 1 white and 1 black ball, of which the chance was $\frac{8}{15}$.

(c) 2 black balls, of which the chance was $\frac{1}{15}$.

Since 1 white ball has been drawn, (c) is impossible; we have, therefore, only (a) and (b) to consider.

Supposing (a) to exist, the chance of drawing a white ball from the second box was 1; supposing (b) to exist, the chance of drawing a white ball from the second box was $\frac{1}{2}$.

Hence, the chance *before the event* that (a) exists, and we draw a white ball, that is, the chance that we draw a white ball from 2 white balls, was $\frac{2}{5} \times 1 = \frac{2}{5}$; the chance *before the event* that (b) exists, and we draw a white ball, that is, the chance that we draw a white ball from a white ball and a black ball, was $\frac{8}{15} \times \frac{1}{2} = \frac{4}{15}$.

Represent by Q_1 the chance *after the event* that (a) existed, and by Q_2 the chance *after the event* that (b) existed.

We shall *assume* that Q_1 and Q_2 are *proportional* to the chance *before the event* that a white ball would be drawn from (a), and the chance *before the event* that a white ball would be drawn from (b).

This assumption corresponds to the assumption in Examples (1) and (2), in which the cases were equally likely to occur. We assume, then, that

$$Q_1 : Q_2 = \tfrac{2}{5} : \tfrac{4}{15}, \text{ or } \frac{Q_1}{\frac{2}{5}} = \frac{Q_2}{\frac{4}{15}}.$$

$$\therefore \frac{Q_1}{\frac{2}{5}} = \frac{Q_2}{\frac{4}{15}} = \frac{Q_1 + Q_2}{\frac{2}{5} + \frac{4}{15}}.$$

But $Q_1 + Q_2 = 1$, since either (a) or (b) *must* exist; also $\frac{2}{5} + \frac{4}{15} = \frac{2}{3}$.

$$\therefore \frac{Q_1}{\frac{2}{5}} = \frac{Q_2}{\frac{4}{15}} = \frac{1}{\frac{2}{3}}.$$

$$\therefore Q_1 = \tfrac{3}{5}, \text{ and } Q_2 = \tfrac{2}{5}.$$

The chance that both balls are white is $\frac{3}{5}$.

370. In general, let P_1, P_2, P_3, $\cdots$ be the chance *before the event* that the first, second, third, $\cdots$ cause exists; and p_1, p_2,

p_3, $\cdots$ the chance *before the event* that, when the first, second, third, $\cdots$ cause exists, the event will follow. Let Q_1, Q_2, Q_3, $\cdots$ be the chance *after the event* that the first, second, third, $\cdots$ cause existed.

Then P_1p_1 is the chance before the event that the event will happen from the first cause; P_2p_2, the chance before the event that the event will happen from the second cause; and so on.

We shall *assume* that Q_1, Q_2, Q_3, $\cdots$ are, respectively, proportional to P_1p_1, P_2p_2, P_3p_3, $\cdots$

That is, $$\frac{Q_1}{P_1p_1} = \frac{Q_2}{P_2p_2} = \frac{Q_3}{P_3p_3} = \cdots$$

Therefore, by § 245,

$$\frac{Q_1}{P_1p_1} = \frac{Q_2}{P_2p_2} = \frac{Q_3}{P_3p_3} = \cdots = \frac{Q_1 + Q_2 + Q_3 + \cdots}{P_1p_1 + P_2p_2 + P_3p_3 + \cdots}.$$

But $Q_1 + Q_2 + Q_3 + \cdots = 1$, since some one of the causes must exist. Hence,

$$\frac{Q_1}{P_1p_1} = \frac{Q_2}{P_2p_2} = \frac{Q_3}{P_3p_3} = \cdots = \frac{1}{P_1p_1 + P_2p_2 + P_3p_3 + \cdots},$$

from which Q_1, Q_2, Q_3, $\cdots$ may readily be found.

Exercise 53

1. An even number greater than 6 has been thrown with 2 dice. What is the chance that doublets were thrown?

2. A number divisible by 3 has been thrown with 2 dice. What is the chance that the number was odd?

3. Fourteen has been thrown with 3 dice. Find the chance that one, and only one, of the dice turned up a six.

4. An even number greater than 10 has been thrown with 3 dice. Find the chance that the number was 14.

5. From a bag containing 6 white and 2 black balls a person draws 3 balls at random and places them in a second bag. A second person then draws from the second bag 2 balls and finds them to be both white. Find the chance that the third ball in the second bag is white.

6. A bag contains 4 balls, each of which is equally likely to be white or black. A person is to receive \$12 if all four are white. Find the value of his expectation.

Suppose he draws 2 balls and finds them to be both white. What is now the value of his expectation?

7. A and B obtain the same answer to a certain problem. It is found that A obtains a correct answer 11 times out of 12, and B 9 times out of 10. If it is 100 to 1 against their making the same mistake, find the chance that the answer they both obtain is correct.

8. From a pack of 52 cards one has been lost; from the imperfect pack 2 cards are drawn and found to be both spades. Required the chance that the missing card is a spade.

371. Expectation of Life. The subjoined table gives the mortality experience of thirty-five life insurance companies. Columns A show the age-year; columns D show the number of deaths during the corresponding age-years in columns A; and columns S show the number who survive at the end of the year; that is, the number who attain the full age in columns A.

Thus, out of 1000 healthy persons who attain the age of 10 years, 4 die at that age, that is, during their 11th year, and 996 survive to attain the full age of 11 years. Again, looking opposite the 31 in column A, we find that of the 1000 persons arriving at the age of 10 years, 7 die during their 31st year and 883 survive to attain the full age of 31. Hence, 890 out of the 1000 must have survived to the full age of 30, and 110 had died without attaining that age.

A	*D*	*S*	*A*	*D*	*S*	*A*	*D*	*S*	*A*	*D*	*S*	*A*	*D*	*S*
11	4	996	29	7	897	47	10	750	65	20	492	83	17	87
12	4	992	30	7	890	48	10	740	66	21	471	84	15	72
13	4	988	31	7	883	49	11	729	67	22	449	85	13	59
14	4	984	32	7	876	50	11	718	68	22	427	86	12	47
15	4	980	33	7	869	51	11	707	69	22	405	87	10	37
16	4	976	34	7	862	52	12	695	70	23	382	88	9	28
17	4	972	35	8	854	53	12	683	71	23	359	89	7	21
18	5	967	36	8	846	54	13	670	72	24	335	90	5.3	15.7
19	5	962	37	8	838	55	13	657	73	25	310	91	4.4	11.3
20	6	956	38	8	830	56	13	644	74	25	285	92	3.3	8.0
21	6	950	39	8	822	57	14	630	75	26	259	93	2.5	5.5
22	6	944	40	8	814	58	15	615	76	25	234	94	1.8	3.7
23	6	938	41	9	805	59	15	600	77	25	209	95	1.3	2.4
24	6	932	42	9	796	60	16	584	78	23	186	96	0.9	1.5
25	7	925	43	9	787	61	17	567	79	23	163	97	0.6	0.9
26	7	918	44	9	778	62	18	549	80	21	142	98	0.4	0.5
27	7	911	45	9	769	63	18	531	81	20	122	99	0.3	0.2
28	7	904	46	9	760	64	19	512	82	18	104	100	0.2	0

(1) What is the chance that a person who has just completed his 51st year dies before he is 52?

Out of every 707 healthy persons who complete the 51st year of their lives, 12 die during their 52d year and 695 survive. Hence, the chance of the death during his 52d year of the person in question is $\frac{12}{707}$.

(2) What is the chance that a person aged 20 lives till he is 50?

Out of every 956 persons who attain the age of 20 years, 718 survive to attain the full age of 50. Hence, the chance that the person in question lives till he is 50 is $\frac{718}{956}$.

(3) What is the expectation of life of a person who has just completed his 90th year?

The chance that he will die during his 91st year is $\frac{44}{157}$, during his 92d year $\frac{33}{157}$, during his 93d year $\frac{25}{157}$, during his 94th year $\frac{18}{157}$, and so on as per table. But if he dies during his 91st year, he may die with

equal probability in any part of it; hence, his expectation of life is $\frac{1}{2}$ year. So if he dies during his 92d year, his expectation will be $1\frac{1}{2}$ years. If he dies during his 93d year, his expectation will be $2\frac{1}{2}$ years, and so on. Hence, his whole expectation will be

$$\frac{44\times 1+33\times 3+25\times 5+18\times 7+13\times 9+9\times 11+6\times 13+4\times 15+3\times 17+2\times 19}{157\times 2}$$

$$=\tfrac{837}{314}=2\tfrac{209}{314} \text{ years of life.}$$

(4) What is the expectation of life of a person who has just completed his 80th year?

The chance that he will die during his 81st year is $\frac{20}{142}$, his 82d year is $\frac{18}{142}$, his 83d year is $\frac{17}{142}$, and so on. His expectation of life prior to the completion of his 90th year is

$$\frac{20\times 1+18\times 3+17\times 5+15\times 7+13\times 9+\cdots+5.3\times 19}{142\times 2}$$

$$=\tfrac{9977}{2840}=3.5.$$

The chance that he will survive his 90th year is $\frac{157}{1420}$. Therefore, his expectation of life subsequent to his 90th year is

$$\tfrac{157}{1420}\left(10+\tfrac{837}{314}\right)=1.4,$$

the 10 years being added to the result of Example (3).

Hence, his whole expectation is $3.5+1.4=4.9$ years.

Exercise 54

1. If B has just attained the age of 21, what is the chance of his death within a year? Within 5 years? Within 10 years?

2. If A is just 25 years old, what is the chance of his living till he is 50? Till he is 60? Till he is 75?

3. B and C are twins just 18 years old. What is the chance that they will both attain the age of 50? That one, but not both, will die before the age of 50?

4. A bridegroom of 24 marries a bride of 21. What is the chance that they will live to celebrate their golden wedding?

5. What is the expectation of life of a person who has attained the age of 75? Of 70? Of 60?

CHAPTER XXIV

VARIABLES AND LIMITS

372. Constants and Variables. A number that, under the conditions of the problem into which it enters, may take *different values* is called a **variable.**

A number that, under the conditions of the problem into which it enters, has a *fixed value* is called a **constant.**

Variables are generally represented by x, y, z, etc.; constants, by the Arabic numerals, and by a, b, c, etc.

373. Functions. Two variables may be so related that a change in the value of one produces a change in the value of the other. In this case the second variable is said to be a **function** of the first.

Thus, if a man walks on a road at a uniform rate of a miles per hour, the number of miles he walks and the number of hours he walks are both variables, and the first is a function of the second. If y is the number of miles he has walked at the end of x hours, y and x are connected by the relation $y = ax$, and y is a function of x. Also, $x = \frac{y}{a}$; hence, x is also a function of y.

When one of two variables is a function of the other, the relation between them is generally expressed by an equation. If any value of the variable is assumed, the corresponding value or values of the function can be found from the given equation.

The variable of which the value is assumed is generally called the *independent* variable; and the function is called the *dependent* variable.

In the last example we may assume values of x, and find the corresponding values of y from the relation $y = ax$; or assume values of y, and find the corresponding values of x from the relation $x = \frac{y}{a}$. In the first case x is the independent variable, and y the dependent; in the second case y is the independent variable, and x the dependent.

374. Limits. As a variable changes its value, it may approach some constant; if the variable can be made to approach the constant *as near as we please*, the variable is said to *approach the constant as a limit*, and the constant is called the **limit** of the variable.

Let x represent the sum of n terms of the infinite series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots$$

Then (§ 276), $\quad x = \dfrac{(\frac{1}{2})^n - 1}{\frac{1}{2} - 1} = \dfrac{2^n - 1}{2^{n-1}} = 2 - \dfrac{1}{2^{n-1}}.$

Suppose n to increase; then, $\dfrac{1}{2^{n-1}}$ decreases, and x approaches 2.

Since we can take as many terms of the series as we please, n can be made as large as we please; therefore, $\dfrac{1}{2^{n-1}}$ can be made as small as we please, and x can be made to approach 2 as near as we please.

If we take any *assigned* positive constant, as $\frac{1}{10000}$, we can make the difference between 2 and x less than this assigned constant; for we have only to take n so large that $\dfrac{1}{2^{n-1}}$ is less than $\dfrac{1}{10000}$; that is, that 2^{n-1} is greater than 10,000: this is accomplished by taking n as large as 15. Similarly, by taking n large enough, we can make the difference between 2 and x less than *any* assigned positive constant.

Since $2 - x$ can be made as small as we please, it follows that the sum of n terms of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, as n is constantly increased, approaches 2 *as a limit*.

375. Test for a Limit. In order to prove that a variable approaches a constant as a limit, it is *necessary* and *sufficient* to prove that the difference in absolute value between the variable and the constant can become and remain *less than any assigned constant, however small.*

A variable may approach a constant without approaching it *as a limit.*

Thus, in the last example x approaches 3, but not as a limit; for $3 - x$ cannot be made as near to 0 as we please, since it cannot be made less than 1.

376. Infinitesimals. As a variable changes its value, it may constantly decrease in absolute value; if the variable can become and remain less in absolute value than any assigned constant *however small,* the variable is said to *decrease without limit,* or to *decrease indefinitely.* In this case the variable approaches zero as a limit.

When a variable that approaches zero as a limit is conceived to become and remain less in absolute value than any assigned constant however small, the variable is said to become *infinitesimal;* such a variable is called an *infinitesimal number,* or simply an **infinitesimal**.

377. Infinites. As a variable changes its value, it may constantly increase in absolute value; if the variable can become greater in absolute value than any assigned constant *however great,* the variable is said to *increase without limit,* or to *increase indefinitely.*

When a variable is conceived to become and remain greater in absolute value than any assigned constant however great, the variable is said to become *infinite;* such a variable is called an *infinite number,* or simply an **infinite**.

Infinites and infinitesimals are *variables,* not constants. There is no idea of *fixed* value implied in either an infinite or an infinitesimal.

A constant whose absolute value can be shown to be less than the absolute value of any assigned constant however small can have no other value than zero.

378. Finites. A number that cannot become an infinite or an infinitesimal is said to be a **finite number**, or simply a **finite**.

379. Relations between Infinites and Infinitesimals.

I. *If* x *is infinitesimal and* a *is finite and not* 0, *then* ax *is infinitesimal.*

For, ax can be made less in absolute value than any assigned constant since x can be made less than any assigned constant.

II. *If* X *is infinite and* a *is finite and not* 0, *then* aX *is infinite.*

For, aX can be made larger in absolute value than any assigned constant however large since X can be made larger in absolute value than any assigned constant however large.

III. *If* x *is infinitesimal and* a *is finite and not* 0, *then* $\frac{a}{x}$ *is infinite.*

For, $\frac{a}{x}$ can be made larger in absolute value than any assigned constant however large since x can be made less in absolute value than any assigned constant however small.

IV. *If* X *is infinite and* a *is finite and not* 0, *then* $\frac{a}{X}$ *is infinitesimal.*

For, $\frac{a}{X}$ can be made less in absolute value than any assigned constant however small since X can be made larger in absolute value than any assigned constant however large.

In the above theorems a *may be a constant or a variable; the only restriction on the value of* a *is that it shall not become either infinite or zero.*

380. From § 197 one root of the quadratic equation $ax^2 + bx + c = 0$ is infinite when a is infinitesimal, and both roots are infinite when a and b are both infinitesimal.

381. Abbreviated Notation. An infinite is often represented by ∞. In § 379, III and IV are sometimes written

$$\frac{a}{0} = \infty, \quad \frac{a}{\infty} = 0.$$

The expression $\frac{a}{0}$ cannot be interpreted literally since we cannot divide by 0; neither can $\frac{a}{\infty}=0$ be interpreted literally since we can find no number such that the quotient obtained by dividing a by that number is zero.

$\frac{a}{0}=\infty$ is simply an abbreviated way of writing: ***if*** **$\frac{a}{x}=X$,** ***and*** **x** ***approaches*** **0** ***as a limit,*** **X** ***increases without limit.***

$\frac{a}{\infty}=0$ is simply an abbreviated way of writing: ***if*** **$\frac{a}{X}=x$** ***and*** **X** ***increases without limit,*** **x** ***approaches*** **0** ***as a limit.***

The symbol $\doteq$ is used for the phrase *approaches as a limit.*

Thus, $x \doteq a$ means and is read ***as*** **x** ***approaches*** **a** ***as a limit.***

382. Approach to a Limit. When a variable approaches a limit it may approach its limit in one of three ways:

1. The variable may be always less than its limit.
2. The variable may be always greater than its limit.
3. The variable may be sometimes less and sometimes greater than its limit.

If x represents the sum of n terms of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$, x is always less than its limit 2.

If x represents the sum of n terms of the series $3-\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\cdots$, x is always greater than its limit 2.

If x represents the sum of n terms of the series $3-\frac{3}{2}+\frac{3}{4}-\frac{3}{8}+\cdots$, we have (§ 276)

$$x=\frac{3-3\left(-\frac{1}{2}\right)^n}{1+\frac{1}{2}}=2-2\left(-\tfrac{1}{2}\right)^n.$$

As n is indefinitely increased, x evidently approaches 2 as a limit.

If n is even, x is less than 2; if n is odd, x is greater than 2. Hence, if n is increased by taking each time one more term, x is alternately less than and greater than 2. If, for example,

$$\begin{array}{llllll} n=2, & 3, & 4, & 5, & 6, & 7, \\ x=1\frac{1}{2}, & 2\frac{1}{4}, & 1\frac{7}{8}, & 2\frac{1}{16}, & 1\frac{31}{32}, & 2\frac{1}{64}. \end{array}$$

In whatever way a variable approaches a constant, the test for a limit given in § 375 always applies.

383. Equal Variables. *If two variables are always equal, and each approaches a limit, then their limits are equal.*

Let x and y be increasing variables, a and b their limits.

Now, $$a = x + x' \text{ and } b = y + y', \qquad (\S\ 375)$$
where x' and y' are variables which approach 0 as a limit.

Then, since the equation $x = y$ always holds true,
$$a - b = x' - y'.$$

But $x' - y'$ can be made less than any assigned constant since x' and y' can each be made less than any assigned constant.

Since $x' - y'$ is always equal to the constant $a - b$, $x' - y'$ must be a constant. But the only constant which is less than any assigned constant is 0. (§ 337)

Therefore, $$x' - y' = 0.$$

Hence, $$a - b = 0, \text{ or } a = b.$$

384. Limit of a Sum. *The limit of the algebraic sum of any finite number of variables is the algebraic sum of their limits.*

Let $x, y, z, \cdots$ be variables, and $a, b, c, \cdots$ their limits. Then, $a - x$, $b - y$, $c - z$, $\cdots$ are variables which can each be made less than any assigned constant (§ 375). Then, $(a - x) + (b - y) + (c - z) + \cdots$ can be made less than any assigned constant.

For, let v be the numerically greatest of the variables $a - x$, $b - y$, $c - z$, $\cdots$, and n the number of variables.

Then, $(a - x) + (b - y) + (c - z) + \cdots < v + v + v \cdots$ to n terms.

But $v + v + v \cdots$ to n terms $= nv$.

Now nv can be made less than any assigned constant since n is finite, and v can be made less than any assigned constant (§ 379, I).

Therefore, $(a - x) + (b - y) + (c - z) + \cdots$, which is less than nv, can be made less than any assigned constant.

$\therefore$ $(a + b + c + \cdots) - (x + y + z + \cdots)$ can be made less than any assigned constant.

$\therefore$ $a + b + c + \cdots$ is the limit of $(x + y + z + \cdots)$. (§ 375)

385. Limit of a Product. *The limit of the product of two or more variables is the product of their limits.*

Let x and y be variables, a and b their limits.

To prove that ab is the limit of xy.

Put $x = a - x'$, $y = b - y'$; then x' and y' are variables that can be made less than any assigned constant. (§ 375)

Now,
$$xy = (a - x')(b - y')$$
$$= ab - ay' - bx' + x'y'.$$
$$\therefore ab - xy = ay' + bx' - x'y'.$$

Since every term on the right contains x' or y', the right member can be made less than any assigned constant. (§ 384)

Hence, $ab - xy$ can be made less than any assigned constant.

Therefore, ab is the limit of xy. (§ 375)

Similarly for three or more variables.

386. Limit of a Quotient. *The limit of the quotient of two variables is the quotient of their limits, if the divisor* $\neq$ *zero.*

Let x and y be variables, a and b their limits.

Put $a - x = x'$, and $b - y = y'$; then x' and y' are variables with limit 0. (§ 375)

We have
$$x = a - x',\ y = b - y',\ \text{and}\ \frac{x}{y} = \frac{a - x'}{b - y'}.$$

Now,
$$\frac{a}{b} - \frac{x}{y} = \frac{a}{b} - \frac{a - x'}{b - y'} = \frac{bx' - ay'}{b(b - y')}.$$

The numerator of the last expression approaches 0 as a limit, and the denominator approaches b^2 as a limit; hence, the expression approaches 0 as a limit. (§ 379, I)

Therefore, $\frac{a}{b} - \frac{x}{y}$ approaches 0 as a limit.

Therefore, $\frac{a}{b}$ is the limit of $\frac{x}{y}$. (§ 375)

387. Vanishing Fractions. When variables are involved in both numerator and denominator of a fraction it may happen that for certain values of the variables the numerator and the denominator both vanish. The fraction then assumes the form $\frac{0}{0}$, a form without meaning; as even the interpretation of § 381 fails, since the numerator is 0. If, however, there is but *one* variable involved, we may obtain a value as follows:

Let x be the variable, and a the value of x for which the fraction assumes the form $\frac{0}{0}$. Give to x a value a little greater than a, as $a + z$; the fraction now has a definite value. Find the limit of this last value as z is indefinitely decreased. This limit is called the **limiting value** of the fraction.

(1) Find the limiting value of $\dfrac{x^2 - a^2}{x - a}$ as $x \doteq a$.

When x has the value a the fraction assumes the form $\frac{0}{0}$.
Put $x = a + z$; the fraction becomes

$$\frac{(a + z)^2 - a^2}{(a + z) - a} = \frac{2\,az + z^2}{z}.$$

Since z is not 0, we divide by z and obtain $2\,a + z$.
As z is indefinitely decreased, this approaches $2\,a$ as a limit.
Hence, $2\,a$ is the limiting value of the fraction as $x \doteq a$.

(2) Find the limiting value of $\dfrac{2x^3 - 4x + 5}{3x^3 + 2x^2 - 1}$ when $x \doteq \infty$.

We have $$\frac{2\,x^3 - 4\,x + 5}{3\,x^3 + 2\,x^2 - 1} = \frac{2 - \frac{4}{x^2} + \frac{5}{x^3}}{3 + \frac{2}{x} - \frac{1}{x^3}}.$$

As x increases indefinitely, $\frac{4}{x^2}$, $\frac{5}{x^3}$, $\frac{2}{x}$, $\frac{1}{x^3}$ approach 0 as a limit (§ 379, IV), and the fraction approaches $\frac{2}{3}$ as a limit.

Exercise 55

Find the limiting value of:

1. $\dfrac{(4x^2-3)(1-2x)}{7x^3-6x+4}$ when x becomes infinitesimal.

2. $\dfrac{(x^2-5)(x^2+7)}{x^4+35}$ when x becomes infinite.

3. $\dfrac{(x+2)^3}{x^2+4}$ when x becomes infinitesimal.

4. $\dfrac{x^2-8x+15}{x^2-7x+12}$ when x approaches 3.

5. $\dfrac{x^2-9}{x^2+9x+18}$ when x approaches -3.

6. $\dfrac{x(x^2+4x+3)}{x^3+3x^2+5x+3}$ when x approaches -1.

7. $\dfrac{x^3+x^2-2}{x^3+2x^2-2x-1}$ when x approaches 1.

8. $\dfrac{4x+\sqrt{x-1}}{2x-\sqrt{x+1}}$ when x approaches 1.

9. $\dfrac{x-1}{\sqrt{x^2-1}+\sqrt{x-1}}$ when x approaches 1.

10. $\dfrac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$ when x approaches 2.

11. $\dfrac{\sqrt{x-a}+\sqrt{x}-\sqrt{a}}{\sqrt{x^2-a^2}}$ when x approaches a.

12. If x approaches a as a limit, and n is a positive integer, show that the limit of x^n is a^n.

13. If x approaches a as a limit, and a is not 0, show that the limit of x^n is a^n, where n is a negative integer.

CHAPTER XXV

SERIES

CONVERGENCY OF SERIES

388. Given Series. A series of numbers is said to be **given** if a law is known by which any term of the series can be calculated when its rank in the series is given.

389. An **infinite series** is a series in which the number of terms may be made greater than any finite number.

Thus, if we divide the numerator of the fraction $\frac{1}{1-x}$ by the denominator, we obtain the series $1 + x + x^2 + x^3 + \cdots$ Since we may carry the division as far as we please, it is evident that we may make the number of terms in the series greater than any finite number.

Hence, $1 + x + x^2 + x^3 + \cdots$ is an infinite series.

390. Convergent Series. An infinite series is a *convergent series* if the limit of the sum of the first n terms, when n increases indefinitely, is a definite finite number.

Thus, if $x < 1$, the series $1 + x + x^2 + x^3 + \cdots$ is an infinite decreasing geometrical series and

$$s = \frac{1}{1-x}. \qquad (\S\ 280)$$

That is, the limit of the sum of the first n terms of the series, when n is made to increase indefinitely, is the definite finite number $\frac{1}{1-x}$, and the series is *convergent*.

Every *finite* series is a convergent series.

391. Sum of Convergent Series. The limit of the sum of the first n terms of an infinite convergent series, when n increases indefinitely, is called the *sum of the series*.

392. Divergent Series. An infinite series is a *divergent series* if the sum of the first n terms may be made greater than any assigned finite number if n is made large enough.

Thus, if $x = 1$ or $x > 1$ in the infinite series $1 + x + x^2 + x^3 + \cdots$, it is evident that by making n large enough we can make the sum of the first n terms greater than any assigned finite number.

Hence, if $x = 1$ or $x > 1$, the series $1 + x + x^2 + x^3 + \cdots$ is *divergent.*

393. Oscillating Series. An infinite series is an *oscillating series* if the sum of the first n terms approaches different finite numbers as n is increased.

Thus, if $x = -1$ in the infinite series $1 + x + x^2 + x^3 + \cdots$, the series becomes $1 - 1 + 1 - 1 + \cdots$ If we take an even number of terms, their sum is 0; if an odd number, their sum is 1.

Hence, if $x = -1$, the series $1 + x + x^2 + x^3 + \cdots$ is *oscillating.*

394. In general, we let $u_1, u_2, u_3, \cdots, u_n, \cdots$ represent any infinite series each of whose terms is finite.

395. Residue of a Series. The difference between the sum of an infinite series and the sum of the first n terms if n increases indefinitely is called the *residue of the series.*

Let S represent the sum of a series,

S_n represent the sum of the first n terms,

and R_n represent the residue after the first n terms.

Then, by the definition of the residue,

$$S - S_n = R_n.$$

396. *If an infinite series is convergent, its residue is an infinitesimal.*

For $$S - S_n = R_n. \qquad (\S\ 395)$$

Since by hypothesis the series is convergent,

$$S = \text{the limit of } S_n. \qquad (\S\ 390)$$

$$\therefore S - S_n \text{ is an infinitesimal.} \qquad (\S\ 375)$$

$$\therefore R_n \text{ is an infinitesimal.} \qquad (\S\ 395)$$

397. ***If an infinite series is convergent, the*** n***th term is an infinitesimal when*** n ***increases indefinitely.***

For $S - S_{n-1}$ is an infinitesimal, (§ 396)

also $S - S_n$ is an infinitesimal.

Hence, $S - S_{n-1} - (S - S_n)$ is an infinitesimal,

or, $S_n - S_{n-1}$ is an infinitesimal.

But $S_n - S_{n-1} = u_n$.

Therefore, u_n is an infinitesimal.

398. ***If an infinite series is convergent,*** m ***can be made so large that the sum of*** p ***consecutive terms beginning with the*** (m + 1)***th will be an infinitesimal, however great*** p ***may be made.***

Let $S =$ the sum of the series,

$S_m =$ the sum of the first m terms,

and $S_{m+p} =$ the sum of the first $m + p$ terms.

Then, $S - S_m =$ the sum of all the terms after the mth,

and $S - S_{m+p} =$ the sum of all the terms after the $(m + p)$th.

Hence, $S - S_m - (S - S_{m+p}) = u_{m+1} + u_{m+2} + \cdots + u_{m+p}$,

or, $$S_{m+p} - S_m = u_{m+1} + u_{m+2} + \cdots + u_{m+p}. \quad [1]$$

Let $$m + p = n. \quad [2]$$

Then [1] becomes $$S_n - S_m = u_{m+1} + u_{m+2} + \cdots + u_n. \quad [3]$$

Now let p increase indefinitely.

Then, by [2] n must also increase indefinitely,

and $$\text{limit } S_n = S.$$

Then [3] becomes

$$S - S_m = \underset{p \doteq \infty}{\text{limit}} (u_{m+1} + u_{m+2} + u_{m+3} + \cdots). \quad [4]$$

Now let m be made to increase indefinitely.

Then, $S - S_m$ is an infinitesimal. (§ 396)

Therefore, $\underset{p \doteq \infty}{\text{limit}} (u_{m+1} + u_{m+2} + \cdots)$ is an infinitesimal.

Therefore, by making m large enough we can make

$$u_{m+1} + u_{m+2} + u_{m+3} + \cdots + u_{m+p}$$

an infinitesimal, however large p may be.

399. *If in an infinite series the sum of* p *consecutive terms beginning with the* (m + 1)*th is an infinitesimal, however great* p *may be made, then the series is convergent.*

Let e represent any positive number taken as small as we please.

Since by hypothesis

$$u_{m+1} + u_{m+2} + u_{m+3} + \cdots + u_{m+p} \qquad [1]$$

is an infinitesimal, then

$$u_{m+1} + u_{m+2} + u_{m+3} + \cdots + u_{m+p} < e. \qquad (\S\ 376)$$

But $\quad u_{m+1} + u_{m+2} + u_{m+3} + \cdots + u_{m+p} = S_{m+p} - S_m.$

$$\therefore S_{m+p} - S_m < e. \qquad [2]$$

Let $\quad m + p = n. \qquad [3]$

Then [2] becomes $\quad S_n - S_m < e. \qquad [4]$

Now let p increase indefinitely.

By [3] n must also increase indefinitely,

and $\quad \text{limit } S_n = S.$

Then [4] becomes $\quad S - S_m < e. \qquad [5]$

Since by [5] $S - S_m$ is an infinitesimal,

$$S = \text{limit } S_m. \qquad (\S\ 375)$$

Therefore, the series is convergent. (§ 390)

400. *If an infinite series is convergent, the residue* R_m *is an infinitesimal.*

For, when p is made to increase indefinitely (§ 399),

$$u_{m+1} + u_{m+2} + u_{m+3} + \cdots + u_{m+p} \text{ is an infinitesimal.}$$

But $\quad u_{m+1} + u_{m+2} + u_{m+3} + \cdots + u_{m+p} = R_m.$

Therefore, R_m is an infinitesimal.

401. The theorems of §§ 399 and 400 may be stated,

If the residue R_m *of an infinite series is an infinitesimal, the series is convergent; and, conversely, if an infinite series is convergent,* R_m *is an infinitesimal.*

402. *If an infinite series has positive terms only and* S_n *remains less than a known finite magnitude* M, *however great* n *may be made, then the series is convergent.*

For, if the series could be divergent,

$$\lim_{n \doteq \infty} S_n = \infty,$$

and it would be possible to make n so great that S_n would be greater than M, which is contrary to the hypothesis.

Therefore, the series is convergent.

403. *If the infinite series* $v_1 + v_2 + v_3 + \cdots$ *has positive terms only and is convergent, and if from a definite term onwards* $u_n = v_n$ *or* $u_n < v_n$, *then the series* $u_1 + u_2 + u_3 + \cdots$ *is also convergent.*

Let $\qquad V_n = v_1 + v_2 + v_3 + \cdots + v_n,$

and $\qquad U_n = u_1 + u_2 + u_3 + \cdots + u_n.$

Since the first series is convergent by hypothesis, we can take m so great that

$$v_{m+1} + v_{m+2} + v_{m+3} + \cdots + v_{m+p} < e,$$

however great p may be made. (§ 398)

But $\quad u_{m+1} \ngtr v_{m+1},\ u_{m+2} \ngtr v_{m+2},\ \cdots,\ u_{m+p} \ngtr v_{m+p}.$

Hence, $\qquad u_{m+1} + u_{m+2} + \cdots + u_{m+p} < e.$

Therefore, the series $u_1 + u_2 + u_3 + \cdots$ is convergent. (§ 399)

404. *If the infinite series* $v_1 + v_2 + v_3 + \cdots$ *has positive terms only and is divergent, and if from a definite term onwards* $u_n = v_n$ *or* $u_n > v_n$, *then the series* $u_1 + u_2 + u_3 + \cdots$ *is also divergent.*

For, if the u series could be convergent, then, since by hypothesis $v_n \not> u_n$, the v series by § 403 would be convergent, which is contrary to the hypothesis.

Therefore, the series $u_1 + u_2 + u_3 + \cdots$ is divergent.

405. *If the infinite series* $u_1 + u_2 + u_3 + \cdots$ *has positive terms only, and if from a given term onwards, say from the* n*th*, $\frac{u_{n+1}}{u_n} \not> k < 1$, *where* k *is a constant independent of* n, *then the series is convergent.*

By hypothesis, after the nth term $\frac{u_{n+1}}{u_n} \not> k$ for every value of n.

Hence, $u_{n+1} \not> u_n k$ for every value of n.

Therefore, we may make the following table:

$$\begin{array}{lll}
u_m & & = u_m, \\
u_{m+1} & & \not> u_m k, \\
u_{m+2} & \not> u_{m+1}k & \not> u_m k^2, \\
u_{m+3} & \not> u_{m+2}k & \not> u_m k^3, \\
\cdot \quad \cdot & \quad \cdot & \cdot \quad \cdot
\end{array}$$

We see that each term of the series

$$u_m + u_{m+1} + u_{m+2} + u_{m+3} + \cdots \qquad [1]$$

is not greater than the corresponding term of the series

$$u_m + u_m k + u_m k^2 + u_m k^3 + \cdots \qquad [2]$$

But series [2] is a geometrical progression whose ratio is k, and, since by hypothesis $k < 1$, the progression is a decreasing geometrical progression, and therefore convergent. (§ 390)

Hence, series [1] is convergent. (§ 403)

Therefore, the series $u_1 + u_2 + u_3 + \cdots$ is also convergent.

406. *If the infinite series* $u_1 + u_2 + u_3 + \cdots$ *has positive terms only, and if from a definite term onwards, say from the* n*th*, $\frac{u_{n+1}}{u_n} \not< 1$, *the series is divergent.*

For all values of n less than m, $u_{m+1} \not< u_m$.

Therefore, we may make the following table:

$$\begin{array}{lll} u_m & & = u_m, \\ u_{m+1} & & \nless u_m, \\ u_{m+2} & \nless u_{m+1} & \nless u_m, \\ u_{m+3} & \nless u_{m+2} & \nless u_m, \\ \cdot & \cdot \quad \cdot & \cdot \end{array}$$

Now the series $u_m + u_m + u_m + u_m$ is divergent. (§ 392)

Hence, the series $u_m + u_{m+1} + u_{m+2} + \cdots$ is divergent. (§ 404)

Hence, the series $u_1 + u_2 + \cdots + u_m + u_{m+1} + u_{m+2} + \cdots$ is also divergent.

407. ***An infinite series that contains both positive and negative terms is convergent if the series consisting of the absolute values of its terms is convergent.***

Let the given series be

$$u_1 + u_2 + u_3 + \cdots + u_\infty, \qquad [1]$$

and let $$S_{m+p} - S_m = u_{m+1} + u_{m+2} + \cdots + u_{m+p}. \qquad [2]$$

Let $|u_n|$ represent the absolute value of u_n, and let

$$\Sigma = |u_1| + |u_2| + |u_3| + \cdots \qquad [3]$$

Then, since series [3] is convergent by hypothesis, by § 398,

$$\Sigma_{m+p} - \Sigma_m = |u_{m+1}| + |u_{m+2}| + \cdots + |u_{m+p}| < e. \qquad [4]$$

Let $$u_{r+1} + u_{r+2} + u_{r+3} + \cdots + u_{r+p} \qquad [5]$$

contain all the positive terms in series [2],

and let $$u_{s+1} + u_{s+2} + u_{s+3} + \cdots + u_{s+p} \qquad [6]$$

contain the absolute values of all the negative terms in series [2].

Since [5] contains only a part of the terms in [4],

$$u_{r+1} + u_{r+2} + u_{r+3} + \cdots + u_{r+p} < e. \qquad [7]$$

Since [6] contains only a part of the terms in [4],

$$u_{s+1} + u_{s+2} + u_{s+3} + \cdots + u_{s+p} < e. \qquad [8]$$

Hence,

$$u_{r+1}+u_{r+2}+\cdots+u_{r+p}-(u_{s+1}+u_{s+2}+\cdots+u_{s+p})<e, \quad [9]$$

or, $$u_{r+1}+u_{r+2}+\cdots+u_{r+p}-u_{s+1}-u_{s+2}-\cdots-u_{s+p}<e. \quad [10]$$

Since [10] contains all the terms in [2],

$$u_{m+1}+u_{m+2}+u_{m+3}+\cdots+u_{m+p}<e.$$

Therefore, $u_1+u_2+u_3+\cdots+u_n$ is convergent. (§ 399)

408. If the absolute values of the terms of a given series form a convergent series, the given series is said to be **absolutely convergent.**

409. Examples. (1) For what values of x is the infinite series $x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\pm\frac{x^n}{n}\mp\cdots$ convergent?

Here, $$r=\frac{u_{n+1}}{u_n}=\left(\frac{n}{n+1}\right)x=\left(1-\frac{1}{n+1}\right)x.$$

As n is indefinitely increased, r approaches x as a limit.

Hence, the series is convergent when x is numerically less than 1 (§§ 407, 405), and divergent when x is numerically greater than 1.

When $x=1$ the series is convergent by § 403.

When $x=-1$ the series becomes

$$-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots\right),$$

and the series is divergent. (§ 392)

(2) For what values of x is the infinite series

$$\frac{x}{1\times 2}+\frac{x^2}{2\times 3}+\frac{x^3}{3\times 4}+\cdots+\frac{x^n}{n(n+1)} \text{ convergent?}$$

Here, $$r=\frac{u_{n+1}}{u_n}=\left(\frac{n}{n+2}\right)x=\left(\frac{1}{1+\frac{2}{n}}\right)x.$$

As n is indefinitely increased, r approaches x as a limit.

If x is numerically less than 1, the series is convergent. (§ 405)

If x is numerically greater than 1, the series is divergent. (§ 406)

If $x=1$, the series is convergent. (§ 405)

If $x=-1$, the series is convergent. (§ 407)

Exercise 56

Determine whether the following infinite series are convergent or divergent:

1. $x - \dfrac{x^3}{\underline{|3}} + \dfrac{x^5}{\underline{|5}} - \dfrac{x^7}{\underline{|7}} + \cdots$

2. $1 - \dfrac{x^2}{\underline{|2}} + \dfrac{x^4}{\underline{|4}} - \dfrac{x^6}{\underline{|6}} + \cdots$

3. $1 + x + \dfrac{x^2}{\underline{|2}} + \dfrac{x^3}{\underline{|3}} + \cdots$

4. $x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$

5. $x + \dfrac{1}{2} \cdot \dfrac{x^3}{3} + \dfrac{1 \cdot 3}{2 \cdot 4} \cdot \dfrac{x^5}{5} + \dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \dfrac{x^7}{7} + \cdots$

FACTORIAL BINOMIAL THEOREM

410. Factorial Notation. The expression $n!^r$, in which r is a positive integer, denotes the product

$$1 \times n \times (n-1) \times (n-2) \times \cdots \times (n-r+2) \times (n-r+1),$$

and is read *factorial* n *of order* r; n is the **primitive** (factor), and r is the **index of the order.** If the primitive and the index of the order are equal, the latter is omitted.

Thus, $n!^n$ is written simply $n!$.

In writing out a factorial as a product, the initial unit-multiplicand is usually omitted, so that the general practice is to write

$$n!^r \equiv n(n-1)(n-2) \cdots (n-r+1),$$

and

$$n! \equiv n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1.$$

However, inserting the initial unit-multiplicand gives at once the interpretation of the zero index, $n!^0 \equiv 1$, and the extension to negative indices,

$$n!^{-r} \equiv \frac{1}{(n+1)(n+2)\cdots(n+r)}.$$

Thus, $6!^3 = 6 \times 5 \times 4 = 120$; and $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

Exercise 57

Find the value of:

1. $7!^2$.
2. $7!^4$.
3. $7!$.
4. $\dfrac{5!^3}{3!}$.
5. $\dfrac{5!}{2!3!}$.
6. $\dfrac{8!}{4!4!}$.
7. $\dfrac{n!^r}{r!}$.
8. $(-2)!^4$.
9. $(-\tfrac{1}{2})!^3$.
10. $(-n)!^r$.
11. $(3\tfrac{1}{2})!$.
12. $(3\tfrac{1}{2})!^5$.
13. $(\tfrac{1}{2})!^3$.
14. $(-1)!^3$.
15. $3!^{-3}$.
16. $0!^{-4}$.
17. $(-5)!^{-3}$.
18. $(\tfrac{1}{2})!^{-3}$.
19. $n!^r$.
20. $(n+1)(n!^r)$.
21. $n(n+1)\{(n-1)!^r\}$.
22. $(n+1)(n+2)(n!)$.

Assuming that Formula [A], § 288, is true and that n is a positive integer, show that:

23. $$(x+y)^n = x^n + \frac{n!^1}{1!}x^{n-1}y + \frac{n!^2}{2!}x^{n-2}y^2 + \cdots + \frac{n!^r}{r!}x^{n-r}y^r + \cdots$$

24. $$(x+y)^n = n!\left\{\frac{x^n}{n!} + \frac{x^{n-1}y}{(n-1)!1!} + \frac{x^{n-2}y^2}{(n-2)!2!} + \cdots + \frac{x^{n-r}y^r}{(n-r)!r!} + \cdots\right\}.$$

25. $$(x+y)^n = x^n + n!^1 0!^{-1}x^{n-1}y^1 + n!^2 0!^{-2}x^{n-2}y^2 + \cdots + n!^r 0!^{-r}x^{n-r}y^r + \cdots$$

411. Show that $\dfrac{n!^r}{r!} + \dfrac{n!^{r-1}}{(r-1)!} = \dfrac{(n+1)!^r}{r!}$.

Now $\dfrac{n!^r}{r!} = \dfrac{(n-r+1)(n!^{r-1})}{r!}$, and $\dfrac{n!^{r-1}}{(r-1)!} = \dfrac{r(n!^{r-1})}{r!}$.

$$\therefore \frac{n!^r}{r!} + \frac{n!^{r-1}}{(r-1)!} = \frac{(n-r+1)(n!^{r-1}) + r(n!^{r-1})}{r!}$$
$$= \frac{(n+1)(n!^{r-1})}{r!} = \frac{(n+1)!^r}{r!}.$$

412. The Factorial Binomial Theorem. If r is a positive integer and m and n any numbers whatever,

$$(m+n)!^{r} = m!^{r} + \frac{r}{1}m!^{r-1}n!^{1} + \frac{r(r-1)}{1\cdot 2}m!^{r-2}n!^{2} + \cdots$$
$$+ \frac{r(r-1)\cdots(r-t+1)}{1\cdot 2\cdot 3\cdots t}m!^{r-t}n!^{t} + \cdots \qquad [1]$$

By successive multiplications we obtain the following identities:

$$(m+n)!^{2} \equiv m!^{2} + 2m!^{1}n!^{1} + n!^{2};$$
$$(m+n)!^{3} \equiv m!^{3} + 3m!^{2}n!^{1} + 3m!^{1}n!^{2} + n!^{3};$$
$$(m+n)!^{4} \equiv m!^{4} + 4m!^{3}n!^{1} + 6m!^{2}n!^{2} + 4m!^{1}n!^{3} + n!^{4}.$$

The proper method of obtaining the expanded forms on the right is as follows:

$$\begin{array}{rlllll}
m+n= & m & +n & & & \\
m+n-1= & \underline{(m-1)+n;} & \underline{m+(n-1)} & & & \\
 & m(m-1) & +mn & & & \\
 & & \underline{+mn} & \underline{+n(n-1)} & & \\
\therefore (m+n)!^{2}= & m!^{2} & +2mn & +n!^{2} & & \text{(i)} \\
m+n-2= & \underline{(m-2)+n;} & \underline{(m-1)+(n-1);} & \underline{m+(n-2)} & & \\
 & m!^{2}(m-2) & +2mn(m-1) & +m(n!^{2}) & & \text{(ii)} \\
 & & \underline{+m!^{2}n} & \underline{+2mn(n-1)} & \underline{+n!^{2}(n-2)} & \text{(iii)} \\
\therefore (m+n)!^{3}= & m!^{3} & +3m!^{2}n!^{1} & +3m!^{1}n!^{2} & +n!^{3} & \text{(iv)}
\end{array}$$

In the preceding multiplication the line (ii) is formed by multiplying the first term of line (i) by $(m-2)$, the second term by $(m-1)$, and the third term by m. Line (iii) is formed by multiplying the first term of line (i) by n, the second term by $(n-1)$, and the third term by $(n-2)$. Hence, line (iv), which is the sum of lines (ii) and (iii), contains the first term of line (i) multiplied by $(m-2)+n$, the second term multiplied by $(m-1)+(n-1)$, and the third term multiplied by $m+(n-2)$. Therefore, line (iv) is equal to line (i) multiplied by $(m+n-2)$.

Continuing this process to form a line (v) by multiplying the first term of line (iv) by $(m-3)$, the second term by $(m-2)$, the third term by $(m-1)$, and the fourth term by m; and a line (vi) by multiplying the first term of line (iv) by n, the second term by $(n-1)$, the third term by $(n-2)$, and the fourth term by $(n-3)$, we have

$$\begin{array}{rlllll}
(m+n)!^3 = & m!^3 & +3m!^2n!^1 & +3m!^1n!^2 & +n!^3 & \text{(iv)} \\
m+n-3 = & \underline{(m-3)+n;} & \underline{(m-2)+(n-1);} & \underline{(m-1)+(n-2);} & \underline{m+(n-3)} & \\
 & m!^4 & +3m!^3n!^1 & +3m!^2n!^2 & +m!^1n!^3 & \text{(v)} \\
 & & +m!^3n!^1 & +3m!^2n!^2 & +3m!^1n!^3+n!^4 & \text{(vi)} \\
\hline
\therefore (m+n)!^4 = & m!^4 & +4m!^3n!^1 & +6m!^2n!^2 & +4m!^1n!^3+n!^4 & \text{(vii)}
\end{array}$$

These expansions may be written in a form better adapted to show the formation of the coefficients of their terms:

$$(m+n)!^2 = m!^2 + \frac{2}{1}m!^1n!^1 + \frac{2\cdot 1}{1\cdot 2}m!^2;$$

$$(m+n)!^3 = m!^3 + \frac{3}{1}m!^2n!^1 + \frac{3\cdot 2}{1\cdot 2}m!^1n!^2 + \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}n!^3;$$

$$(m+n)!^4 = m!^4 + \frac{4}{1}m!^3n!^1 + \frac{4\cdot 3}{1\cdot 2}m!^2n!^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}m!^1n!^3 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}n!^4.$$

Comparing these expansions with those of the powers of $(a+b)$ as given in § 285, we observe that corresponding terms up to the fourth order and the fourth power have the same coefficients and have the same indices of order as exponents.

To prove that the corresponding coefficients and indices of order are the same as the coefficients and exponents in the expansion of the Binomial Theorem for all positive integral values of index of order and exponent, we proceed exactly as in § 288. We assume that laws 1, 2, and 3 (§ 286) hold true up to a given value, r, of the index of order, and prove that in such case they hold true for the value $r+1$ of the index of order.

Let it be granted that

$$(m+n)!^{r} = m!^{r} + \frac{r}{1}m!^{r-1}n!^{1} + \frac{r(r-1)}{1\cdot 2}m!^{r-2}n!^{2} + \cdots + \frac{r!^{t}}{t!}m!^{r-t}n!^{t} + \cdots \quad [1]$$

Multiply the first term on the right of [1] by $(m-r)$, the second term by $(m-r+1)$, the third term by $(m-r+2)$, and so on; writing the partial products in order in a line.

Form a second line by multiplying the first term on the right of [1] by n, the second term by $(n-1)$, the third term by $(n-2)$, and so on; writing the first partial product of this line under the second partial product of the first line.

Add the two lines of partial products, simplifying the coefficients of the sum by § 411, and we obtain the product of the right member of [1] by $(m+n-r)$. Thus:

$$\begin{array}{llllll}
m!^{r} & +\frac{r}{1}m!^{r-1}n!^{1} & +\frac{r(r-1)}{1\cdot 2}m!^{r-2}n!^{2} & +\cdots+ & \frac{r!^{t}}{t!}m!^{r-t}n!^{t} & +\cdots \\
\hline
(m-r)+n; & (m-r+1)+(n-1); & (m-r+2)+(n-2); & & (m-r+t)+(n-t) & \\
\hline
m!^{r+1}+\frac{r}{1}m!^{r}n!^{1} & +\frac{r(r-1)}{1\cdot 2}m!^{r-1}n!^{2} & +\cdots+\frac{r!^{t}}{t!}m!^{r+1-t}n!^{t} & & & +\cdots \\
\quad +m!^{r}n!^{1} & +\frac{r}{1}m!^{r-1}n!^{2} & +\cdots+\frac{r!^{t-1}}{(t-1)!}m!^{r+1-t}n!^{t} & & & +\cdots \\
\hline
\end{array}$$

$$m!^{r+1} + \frac{r+1}{1}m!^{r}n!^{1} + \frac{(r+1)r}{1\cdot 2}m!^{r-1}n!^{2} + \cdots + \frac{(r+1)!^{t}}{t!}m!^{(r+1)-t}n!^{t} + \cdots$$
$$= (m+n)!^{r+1}. \quad [2]$$

Hence, if the expansion [1] is true for any given positive integer r, it is true for $(r+1)$. Now expansion [1] is true for $r=4$, as shown on page 319. Therefore, it is true for $r=5$; and, being true for $r=5$, it is true for $r=6$; and so on. In short, Formula [1] is true for all positive integral values of the index of order.

Hence, for all positive integral values of r,

$$\mathbf{(m+n)!^{r} = m!^{r} + rm!^{r-1}n!^{1} + \frac{r(r-1)}{1\cdot 2}m!^{r-2}n!^{2} + \cdots + \frac{r!^{t}}{t!}m!^{r-t}n!^{t} + \cdots}$$

NOTE. Theorem [1], § 412, is often named Vandermonde's Factorial Theorem, and Theorem [A], § 288, Newton's Binomial Theorem.

BINOMIAL THEOREM

413. Binomial Theorem; Any Exponent. Let m and n be any two scalar numbers and let the value of x be so taken as to render convergent each of the three series

$$1+\frac{m}{1}x+\frac{m(m-1)}{1\cdot 2}x^2+\cdots+\frac{m!^t}{t!}x^t+\cdots, \tag{i}$$

$$1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\cdots+\frac{n!^t}{t!}x^t+\cdots, \tag{ii}$$

$$1+\frac{m+n}{1}x+\frac{(m+n)(m+n-1)}{1\cdot 2}x^2\cdots+\frac{(m+n)!^t}{t!}x^t\cdots \tag{iii}$$

Then, series (iii) is the product of series (i) and series (ii).

For, on forming the product of series (i) and series (ii) and arranging it according to descending powers of x, the coefficient of x^t is

$$\begin{aligned}
&\frac{m!^t}{t!}+\frac{m!^{t-1}n}{(t-1)!1!^1}+\frac{m!^{t-2}n!^2}{(t-2)!2!}+\frac{m!^{t-3}n!^3}{(t-3)!3!}+\cdots \\
&\equiv\frac{1}{t!}\left\{m!^t+\frac{t}{1}m!^{t-1}n!^1+\frac{t(t-1)}{1\cdot 2}m!^{t-2}n!^2\right. \\
&\qquad\qquad\left.+\frac{t(t-1)(t-2)}{1\cdot 2\cdot 3}m!^{t-3}n!^3+\cdots\right\} \\
&\equiv\frac{(m+n)!^t}{t!}. \qquad (\S\ 412)
\end{aligned}$$

Hence, if x is so taken as to make all three series convergent, we may write

$$\left\{1+\frac{m}{1}x+\frac{m(m-1)}{1\cdot 2}x^2+\cdots\right\}\left\{1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\cdots\right\}$$
$$=1+\frac{m+n}{1}x+\frac{(m+n)(m+n-1)}{1\cdot 2}x^2+\cdots \quad [1]$$

414. If m is any positive integer, then by § 291 series (i) is equal to $(1+x)^m$. If $n=-m$, so that $n+m=0$, Formula [1], § 413, becomes

$$(1+x)^m\left\{1+\frac{-m}{1}x+\frac{-m(-m-1)}{1\cdot 2}x^2\right.$$
$$\left.+\frac{-m(-m-1)(-m-2)}{1\cdot 2\cdot 3}x^3+\cdots\right\}=1.$$

Divide by $(1+x)^m$,

$$1+\frac{-m}{1}x+\frac{-m(-m-1)}{1\cdot 2}x^2+\cdots=\frac{1}{(1+x)^m}=(1+x)^{-m}.$$

$$\therefore (1+x)^{-m}=1+\frac{-m}{1}x+\frac{-m(-m-1)}{1\cdot 2}x^2$$
$$+\frac{-m(-m-1)(-m-2)}{1\cdot 2\cdot 3}x^3+\cdots \quad [2]$$

Comparing this theorem with Formula [A], § 288, we see that [2] merely extends [A] to all negative integral exponents.

415. Let $m=n$ in [1], § 413. Then,

$$\left\{1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\cdots\right\}^2=1+\frac{2n}{1}x+\frac{2n(2n-1)}{1\cdot 2}x^2+\cdots$$

Multiply by $\quad 1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\cdots,$

and reduce the resulting right-hand member by [1], § 413.

Then,

$$\left\{1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\cdots\right\}^3=1+\frac{3n}{1}x+\frac{3n(3n-1)}{1\cdot 2}x^2+\cdots$$

Multiply again by $1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\cdots,$

reduce by [1], § 413, and repeat to q factors, q being any positive integer. Then,

$$\left\{1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\cdots\right\}^q=1+\frac{qn}{1}x+\frac{qn(qn-1)}{1\cdot 2}x^2+\cdots$$

Let $n=\frac{p}{q}$, in which p is any integer, positive or negative. Then,

$$\left\{1+\frac{\frac{p}{q}}{1}x+\frac{\frac{p}{q}\left(\frac{p}{q}-1\right)}{1\cdot 2}x^2+\cdots\right\}^q$$

$$=1+\frac{p}{1}x+\frac{p(p-1)}{1\cdot 2}x^2+\cdots=(1+x)^p$$

by Formula [A], § 288, or Formula [2], § 414, according as p is positive or negative.

Take the arithmetical qth root of each side. Then,

$$(1+x)^{\frac{p}{q}}=1+\frac{\frac{p}{q}}{1}x+\frac{\frac{p}{q}\left(\frac{p}{q}-1\right)}{1\cdot 2}x^2+\frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}-2\right)}{1\cdot 2\cdot 3}x^3+\cdots \qquad [3]$$

If p is prime to q, $(1+x)^{\frac{p}{q}}$ has q different values, and series [3] gives the arithmetical or principal value.

416. On comparing [A], § 288, [2], § 414, and [3], § 415, it will be seen that [2] and [3] are in form included in [A].

Hence, for any rational scalar value of n,

$$(1+x)^n=1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\cdots \qquad [A]$$

provided x is so taken as to render the series of the expansion convergent.

For irrational scalar values of n we may substitute approximate rational values, carrying the approximation to any required degree; or we may carry the approximation closer than any assigned difference, however small in absolute value, and thus prove that Formula [A] holds true for all scalar values of n.

417. If n is fractional or negative, the expansion of $(a+b)^n$ must be in the form $a^n\left(1+\frac{b}{a}\right)^n$ if $a>b$; and in the form $b^n\left(1+\frac{a}{b}\right)^n$ if $b>a$.

418. Convergency of the Binomial Series. In the expansion of $(1+x)^n$, the ratio of the $(r+1)$th term to the rth term is (§ 294)

$$\frac{n-r+1}{r}x, \quad \text{or} \quad \left(\frac{n+1}{r}-1\right)x.$$

If x is positive, and r greater than $n+1$, $\frac{n+1}{r}-1$ is negative. Hence, the terms in which r is greater than $n+1$ are alternately positive and negative.

If x is negative, the terms in which r is greater than $n+1$ are all positive. In either case we have

$$\frac{u_{r+1}}{u_r}=\left(\frac{n+1}{r}-1\right)x;$$

as r is indefinitely increased, this approaches the limit $-x$. Hence (§ 405), the series is convergent if x is numerically less than 1.

419. Examples. (1) Expand $(1+x)^{\frac{1}{3}}$.

$$(1+x)^{\frac{1}{3}}=1+\tfrac{1}{3}x+\frac{\frac{1}{3}(\frac{1}{3}-1)}{1\cdot 2}x^2+\frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{1\cdot 2\cdot 3}x^3+\cdots$$
$$=1+\tfrac{1}{3}x-\frac{2}{3\cdot 6}x^2+\frac{2\cdot 5}{3\cdot 6\cdot 9}x^3-\cdots$$

The above equation is true only for those values of x that make the series convergent.

(2) Expand $\dfrac{1}{\sqrt[4]{1-x}}$.

$$\frac{1}{\sqrt[4]{1-x}}=(1-x)^{-\frac{1}{4}}=1-(-\tfrac{1}{4})x+\frac{-\frac{1}{4}\cdot-\frac{5}{4}}{1\cdot 2}x^2-\frac{-\frac{1}{4}\cdot-\frac{5}{4}\cdot-\frac{9}{4}}{1\cdot 2\cdot 3}x^3+\cdots$$
$$=1+\tfrac{1}{4}x+\frac{1\cdot 5}{4\cdot 8}x^2+\frac{1\cdot 5\cdot 9}{4\cdot 8\cdot 12}x^3+\cdots,$$

if x is so taken that the series is convergent.

A root may often be extracted by means of an expansion.

(3) Extract the cube root of 344 to six decimal places.

$$344 = 343\left(1 + \tfrac{1}{343}\right) = 7^3\left(1 + \tfrac{1}{343}\right).$$

$$\therefore \sqrt[3]{344} = 7\left(1 + \tfrac{1}{343}\right)^{\frac{1}{3}}$$

$$= 7\left[1 + \frac{1}{3}\left(\frac{1}{343}\right) + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{1 \cdot 2}\left(\frac{1}{343}\right)^2 + \cdots\right]$$

$$= 7\,(1 + 0.000971817 - 0.000000944)$$

$$= 7.006796.$$

Exercise 58

Expand to four terms:

1. $(1 + x)^{\frac{1}{2}}$.
2. $(1 + x)^{\frac{1}{4}}$.
3. $\dfrac{1}{\sqrt[3]{1 - x}}$.
4. $(1 - x)^{-4}$.
5. $(1 + x)^{\frac{3}{2}}$.
6. $\dfrac{1}{\sqrt[4]{a^2 - x^2}}$.
7. $\sqrt[5]{2 - 3x}$.
8. $\sqrt[3]{(2 - x^2)^2}$.
9. $\dfrac{1}{\sqrt[4]{(1 + 2x^2)^3}}$.

Find:

10. The eighth term of $(1 - 2x)^{\frac{1}{2}}$.
11. The tenth term of $(a - 3x)^{-\frac{2}{3}}$.
12. The $(r + 1)$th term of $(a + x)^{\frac{1}{3}}$.
13. The $(r + 1)$th term of $(a^2 - 4x^2)^{-\frac{3}{2}}$.
14. The square root of 65 to five decimal places.
15. The seventh root of 129 to six decimal places.
16. Expand $(1 - 2x + 3x^2)^{-\frac{3}{2}}$ to four terms.
17. Find the coefficient of x^4 in the expansion of $\dfrac{(1 + 2x)^2}{(1 + 3x)^3}$.
18. By means of the expansion of $(1 + x)^{\frac{1}{2}}$ show that $\sqrt{2}$ is the limit of the series

$$1 + \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1 \cdot 3}{2 \cdot 3 \cdot 2^3} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 2^4} + \cdots$$

19. Find the first negative term in the expansion of $(1+x)^{\frac{13}{2}}$.

20. Expand $\sqrt{\dfrac{1+x}{1-x}}$ in ascending powers of x to six terms.

21. If n is a positive integer, show that the coefficient of x^{n-1} in the expansion of $(1-x)^{-n}$ is always twice the coefficient of x^{n-2}.

22. If m and n are positive integers, show that the coefficient of x^m in the expansion of $(1-x)^{-n-1}$ is the same as the coefficient of x^n in the expansion of $(1-x)^{-m-1}$.

23. Find the coefficient of x^{2r} in the expansion of $\sqrt{\dfrac{1-x}{1+x}}$ in ascending powers of x.

24. Prove that the coefficient of x^r in the expansion of $(1-4x)^{-\frac{1}{2}}$ is $\dfrac{1\cdot 2\cdot 3\cdots 2r}{(1\cdot 2\cdot 3\cdots r)^2}$.

SERIES OF DIFFERENCES

420. Definitions. If, in any series, we subtract the first term from the second, the second term from the third, and so on, we obtain a first series of differences; in like manner, from this last series we may obtain a second series of differences; and so on. In an arithmetical series the second differences all vanish.

There are series, allied to arithmetical series, in which not the first, but the second, or third, etc., differences vanish.

Thus, take the series

$$
\begin{array}{lccccccccccccccc}
 & 1 & & 5 & & 12 & & 24 & & 43 & & 71 & & 110 & & \cdots \\
\text{First differences,} & & 4 & & 7 & & 12 & & 19 & & 28 & & 39 & & \cdots & \\
\text{Second differences,} & & & 3 & & 5 & & 7 & & 9 & & 11 & & \cdots & & \\
\text{Third differences,} & & & & 2 & & 2 & & 2 & & 2 & & \cdots & & & \\
\text{Fourth differences,} & & & & & 0 & & 0 & & 0 & & \cdots & & & &
\end{array}
$$

such a series, we have

$$\begin{array}{cccccc} a_3 & a_4 & a_5 & a_6 & a_7 & \cdots \\ b_3 & b_4 & b_5 & b_6 & \cdots & \\ c_2 & c_3 & c_4 & c_5 & \cdots & \\ d_2 & d_3 & d_4 & \cdots & & \\ e_1 & e_2 & e_3 & \cdots & & \end{array}$$

ices which all vanish.

a. For simplicity let us take a eries of differences vanishes. Any ed in a manner precisely similar. ich the successive series are formed

$$a_3 = a_2 + b_2 = a_1 + 2\,b_1 + c_1$$
$$b_3 = b_2 + c_2 = b_1 + 2\,c_1 + d_1$$
$$c_3 = c_2 + d_2 = c_1 + 2\,d_1 + e_1$$
$$d_3 = d_2 + e_2 = d_1 + 2\,e_1$$
$$e_3 = e_2 = e_1$$

$$+ b_3 = a_1 + 3\,b_1 + 3\,c_1 + d_1$$
$$+ c_3 = b_1 + 3\,c_1 + 3\,d_1 + e_1$$
$$+ d_3 = c_1 + 3\,d_1 + 3\,e_1$$
$$d_3 + e_3 = d_1 + 3\,e_1$$

$$= a_4 + b_4 = a_1 + 4\,b_1 + 6\,c_1 + 4\,d_1 + e_1$$
$$= b_4 + c_4 = b_1 + 4\,c_1 + 6\,d_1 + 4\,e_1$$
$$c_5 = c_4 + d_4 = c_1 + 4\,d_1 + 6\,e_1$$

$$a_6 = a \qquad + 10\,c_1 + 10\,d_1 + 5\,e_1$$
$$b_6 = \qquad + 10\,d_1 + 10\,e_1$$

$$a_7 \qquad + 15\,c_1 + 20\,d_1 + 15\,e_1$$

so on.

The student will observe that the coefficients in the expression for a_5 are those of the expansion of $(x+y)^4$, and similarly for a_6 and a_7. Hence, in general, if we represent a_1, b_1, c_1, etc., by a, b, c, etc., and put a_{n+1} for the $(n+1)$th term, we obtain the formula

$$a_{n+1} = a + nb + \frac{n(n-1)}{1 \times 2}c + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}d + \cdots$$

Find the 11th term of 1, 5, 12, 24, 43, 71, 110, $\cdots$

Here (§ 420), $a = 1$, $b = 4$, $c = 3$, $d = 2$, $e = 0$; and $n = 10$.

$$\therefore a_{11} = a + 10b + 45c + 120d$$
$$= 1 + 40 + 135 + 240 = 416.$$

422. Sum of the Series. Form a new series of which the first term is 0, and the first series of differences $a_1, a_2, a_3 \cdots$ This series is the following:

$$0,\ a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3,\ a_1 + a_2 + a_3 + a_4, \cdots$$

The $(n+1)$th term of this series is the sum of n terms of the series $a_1, a_2, a_3 \cdots$

(1) Find the sum of 11 terms of the series 1, 5, 12, 24, 43, 71, $\cdots$

The new series is	0		1		6		18		42		85		156
First differences,		1		5		12		24		43		71	
Second differences,			4		7		12		19		28		
Third differences,				3		5		7		9			
Fourth differences,					2		2		2				

Here, $a = 0$, $b = 1$, $c = 4$, $d = 3$, $e = 2$; and $n = 11$.

$$\therefore s = a + 11b + 55c + 165d + 330e$$
$$= 11 + 220 + 495 + 660$$
$$= 1386.$$

If s is the sum of n terms of the series $a_1, a_2, a_3 \cdots$

$$s = 0 + na + \frac{n(n-1)}{1 \times 2}b + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}c + \cdots$$

(2) Find the sum of the squares of the first n natural numbers, $1^2, 2^2, 3^2, 4^2, \cdots, n^2$.

Given series,	1		4		9		16		25		$\cdots$		n^2
First differences,		3		5		7		9		$\cdots$			
Second differences,			2		2		2		$\cdots$				
Third differences,				0		0		$\cdots$					

Therefore, $a = 1$, $b = 3$, $c = 2$, $d = 0$.
These values substituted in the general formula give

$$s = n + \frac{n(n-1)}{1 \times 2} \times 3 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3} \times 2$$
$$= n\left\{1 + \frac{3n}{2} - \frac{3}{2} + \frac{1}{3}(n^2 - 3n + 2)\right\}$$
$$= \frac{n}{6}\{6 + 9n - 9 + 2n^2 - 6n + 4\}$$
$$= \frac{n}{6}\{2n^2 + 3n + 1\} = \frac{n(n+1)(2n+1)}{6}.$$

423. Piles of Spherical Shot. I. When the pile is in the form of a triangular pyramid the summit consists of a single shot resting on three below; and these three rest on a course of six; and these six on a course of ten; and so on, so that the courses form the series

$$1, 1+2, 1+2+3, 1+2+3+4, \cdots, 1+2+\cdots+n.$$

Given series,	1		3		6		10		15		$\cdots$
First differences,		2		3		4		5		$\cdots$	
Second differences,			1		1		1		$\cdots$		
Third differences,				0		0		$\cdots$			

Here, $a = 1$, $b = 2$, $c = 1$, $d = 0$.

These values substituted in the general formula give

$$s = n + \frac{n(n-1)}{2} \times 2 + \frac{n(n-1)(n-2)}{2 \times 3}$$
$$= \frac{n^3 + 3n^2 + 2n}{6}$$
$$= \frac{n(n+1)(n+2)}{1 \times 2 \times 3}.$$

in which n is the number of balls in the side of the bottom course, or the number of courses.

II. When the pile is in the form of a pyramid with a square base the summit consists of one shot, the next course consists of four balls, the next of nine, and so on. Therefore the number of shot is

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2.$$

This sum is $$\frac{n(n+1)(2n+1)}{1 \times 2 \times 3}, \qquad (§ 422)$$

in which n is the number of balls in the side of the bottom course, or the number of courses.

III. When the pile has a base which is rectangular, but not square, the pile terminates with a single row. Suppose p the number of shot in this row; then the second course consists of $2(p+1)$ shot; the third course of $3(p+2)$; and the nth course of $n(p+n-1)$. Hence, the series is

$$p,\ 2p+2,\ 3p+6,\ \cdots,\ n(p+n-1).$$

Given series,	p		$2p+2$		$3p+6$		$4p+12$		$\cdots$
First differences,		$p+2$		$p+4$		$p+6$		$\cdots$	
Second differences,			2		2		$\cdots$		
Third differences,				0		$\cdots$			

Here, $a = p$, $b = p + 2$, $c = 2$, $d = 0$.

These values substituted in the general formula give

$$s = np + \frac{n(n-1)}{2}(p+2) + \frac{n(n-1)(n-2)}{1 \times 2 \times 3} \times 2$$

$$= \frac{n}{6}\{6p + 3(n-1)(p+2) + 2(n-1)(n-2)\}$$

$$= \frac{n}{6}(6p + 3np - 3p + 6n - 6 + 2n^2 - 6n + 4)$$

$$= \frac{n}{6}(3np + 3p + 2n^2 - 2)$$

$$= \frac{n}{6}(n+1)(3p + 2n - 2).$$

If n' denotes the number in the longest row, then $n' = p + n - 1$, and therefore $p = n' - n + 1$. The formula may then be written

$$s = \frac{n}{6}(n+1)(3n' - n + 1),$$

in which n denotes the number of shot in the width, and n' in the length, of the bottom course.

When the pile is incomplete compute the number in the pile as if complete, then the number in that part of the pile which is lacking, and take the difference of the results.

Exercise 59

1. Find the fiftieth term of 1, 3, 8, 20, 43, $\cdots$

2. Find the sum of the series 4, 12, 29, 55, $\cdots$ to 20 terms.

3. Find the twelfth term of 4, 11, 28, 55, 92, $\cdots$

4. Find the sum of the series 43, 27, 14, 4, -3, $\cdots$ to 12 terms.

5. Find the seventh term of 1, 1.235, 1.471, 1.708, $\cdots$

6. Find the sum of the series 70, 66, 62.3, 58.9, $\cdots$ to 15 terms.

7. Find the eleventh term of 343, 337, 326, 310, $\cdots$

8. Find the sum of the series 7×13, 6×11, 5×9, $\cdots$ to 9 terms.

9. Find the sum of n terms of the series 3×8, 6×11, 9×14, 12×17, $\cdots$

10. Find the sum of n terms of the series 1, 6, 15, 28, 45, $\cdots$

11. Show that the sum of the cubes of the first n natural numbers is the square of the sum of the numbers.

12. Determine the number of shot in a side of the base of a triangular pile which contains 286 shot.

13. The number of shot in the top course of a square pile is 169, and in the lowest course 1089. How many shot are there in the pile?

14. Find the number of shot in a rectangular pile having 17 shot in one side of the base and 42 in the other.

15. Find the number of shot in the five lowest courses of a triangular pile which has 15 in one side of the base.

16. The number of shot in a triangular pile is to the number in a square pile, of the same number of courses, as 22 to 41. Find the number of shot in each pile.

17. Find the number of shot required to complete a rectangular pile that has 15 and 6 shot respectively in the sides of its top course.

18. How many shot must there be in the lowest course of a triangular pile that 10 courses of the pile, beginning at the base, may contain 37,020 shot?

19. Find the number of shot in a complete rectangular pile of 15 courses which has 20 shot in the longest side of its base.

20. Find the number of shot in the bottom row of a square pile that contains 2600 more shot than a triangular pile of the same number of courses.

21. Find the number of shot in a complete square pile in which the number of shot in the base and the number in the fifth course above differ by 225.

22. Find the number of shot in a rectangular pile that has 600 in the lowest course and 11 in the top row.

COMPOUND SERIES

424. A **compound series** is a series in which the terms are the sum or the difference of the terms of two other series.

(1) Find the sum of the series

$$\frac{1}{1\times 2},\ \frac{1}{2\times 3},\ \frac{1}{3\times 4},\ \cdots,\ \frac{1}{n(n+1)}.$$

Each term of this series may evidently be expressed in two parts:

$$\frac{1}{1}-\frac{1}{2},\ \frac{1}{2}-\frac{1}{3},\ \cdots,\ \frac{1}{n}-\frac{1}{n+1}.$$

Hence, the sum is

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right),$$

in which the second part of each term except the last is canceled by the first part of the next succeeding term.

Hence, the sum is $1-\dfrac{1}{n+1}$.

As n increases without limit, the sum approaches 1 as a limit.

(2) Find the sum of the series

$$\frac{1}{3\times 5},\ \frac{1}{4\times 6},\ \frac{1}{5\times 7},\ \cdots,\ \frac{1}{n(n+2)}.$$

The terms may be written,

$$\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right),\ \frac{1}{2}\left(\frac{1}{4}-\frac{1}{6}\right),\ \cdots,\ \frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+2}\right).$$

$$\therefore\ \text{Sum}=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{n}-\frac{1}{5}-\frac{1}{6}-\cdots-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}\right)$$

$$=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}-\frac{1}{n+1}-\frac{1}{n+2}\right)$$

$$=\frac{7}{24}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}.$$

As n increases without limit, this sum approaches $\frac{7}{24}$ as a limit.

Exercise 60

Write the general term, and the sum to n terms, and to an infinite number of terms, of the following series:

1. $\dfrac{1}{1\times 4}+\dfrac{1}{2\times 5}+\dfrac{1}{3\times 6}+\cdots$

3. $\dfrac{1}{1\times 5}+\dfrac{1}{5\times 9}+\dfrac{1}{9\times 13}+\cdots$

2. $\dfrac{1}{1\times 3}+\dfrac{1}{2\times 4}+\dfrac{1}{3\times 5}+\cdots$

4. $\dfrac{6}{2\times 7}+\dfrac{6}{7\times 12}+\dfrac{6}{12\times 17}+\cdots$

5. $\dfrac{1}{5\times 11}+\dfrac{1}{8\times 14}+\dfrac{1}{11\times 17}+\cdots$

6. $\dfrac{1}{3\times 8}+\dfrac{1}{6\times 12}+\dfrac{1}{9\times 16}+\cdots$

Write the series of which the general term is:

7. $\dfrac{1}{n(n+1)(n+2)}$.

8. $\dfrac{3n+1}{(n+1)(n+2)(n+3)}$.

INDETERMINATE COEFFICIENTS

425. *If two series which are arranged by powers of* x *are equal for all values of* x *that make both series convergent, the corresponding coefficients are equal each to each.*

Let the equation

$$a + bx + cx^2 + dx^3 + \cdots = A + Bx + Cx^2 + Dx^3 + \cdots \quad [1]$$

hold true for all values of x that make both series convergent.

Since this equation holds true for all values of x which make both series convergent, it holds true when $x = 0$.

For $x = 0$, $\qquad a = A.$ [2]

Subtract [2] from [1], and since for any value of x that is not 0 we may divide by x, divide each member by x; then

$$b + cx + dx^2 + \cdots = B + Cx + Dx^2 + \cdots \quad [3]$$

Then for $x = 0$, $\qquad b = B.$ [4]

In like manner, $\qquad c = C$; and so on.

Hence, the corresponding coefficients are equal each to each.

426. Partial Fractions. To resolve a fraction into *partial fractions* is to express it as the sum of a number of fractions of which the respective denominators are the factors of the denominator of the given fraction. This process is the reverse of the process of *adding* fractions that have different denominators.

Resolution into partial fractions may be easily accomplished by the use of indeterminate coefficients and the theorem of § 425.

In decomposing a given fraction into its simplest partial fractions, it is important to determine what form the assumed fractions must have.

Since the given fraction is the *sum* of the required partial fractions, each assumed denominator must be a factor of the given denominator.

1. All the factors of the given denominator may be real and different.

In this case we take each factor of the given denominator as a denominator of one of the assumed fractions.

Thus,
$$\frac{3x-7}{(x-2)(x-3)}=\frac{A}{x-2}+\frac{B}{x-3}.$$

2. All the factors of the given denominator may be equal.

In this case we assume as denominators every power of the repeated factor from the given power down to the first.

Thus,
$$\frac{x^2+1}{(x-1)^3}=\frac{A}{(x-1)^3}+\frac{B}{(x-1)^2}+\frac{C}{x-1}.$$

3. All the factors may be real and some equal.

In this case we combine the methods of the first two cases.

Thus,
$$\frac{4x^3-63x^2+338x-619}{(x-5)^3(x-7)}=\frac{A}{(x-5)^3}+\frac{B}{(x-5)^2}+\frac{C}{x-5}+\frac{D}{x-7}.$$

4. All the factors may be imaginary.

The imaginary factors occur in pairs of conjugate imaginaries so that the product of each pair is a real quadratic factor.

For example, in the fraction $\dfrac{7x^3-6x^2+9x+108}{(x^2-4x+13)(x^2+2x+5)}$

the factor $x^2+2x+5=(\overline{x+1}-2\sqrt{-1})(\overline{x+1}+2\sqrt{-1})$,

and the factor $x^2-4x+13=(\overline{x+2}-3\sqrt{-1})(\overline{x+2}+3\sqrt{-1})$.

In this case we assume a fraction of the form $\dfrac{Ax+B}{x^2\pm ax+b}$ for each quadratic factor in the given denominator.

Thus,
$$\frac{7x^3-6x^2+9x+108}{(x^2-4x+13)(x^2+2x+5)}=\frac{Ax+B}{x^2-4x+13}+\frac{Cx+D}{x^2+2x+5}.$$

5. Some of the factors may be imaginary.

In this case we combine the method of the fourth case with the method of one of the preceding cases.

Thus,
$$\frac{13x^2-68x+95}{(x-5)(x^2-6x+13)}=\frac{A}{x-5}+\frac{Bx+C}{x^2-6x+13}.$$

(1) Resolve $\dfrac{3x-7}{(x-2)(x-3)}$ into partial fractions.

Assume $$\frac{3x-7}{(x-2)(x-3)} \equiv \frac{A}{x-2} + \frac{B}{x-3}.$$

Then, $$3x-7 \equiv A(x-3) + B(x-2).$$

$$\therefore A + B = 3 \text{ and } 3A + 2B = 7; \qquad (\S\ 425)$$

whence, $$A = 1 \text{ and } B = 2.$$

Therefore, $$\frac{3x-7}{(x-2)(x-3)} \equiv \frac{1}{x-2} + \frac{2}{x-3}.$$

This identity may be verified by actual multiplication.

(2) Resolve $\dfrac{3}{x^3+1}$ into partial fractions.

The denominators will be $x+1$ and x^2-x+1.

Assume $$\frac{3}{x^3+1} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}.$$

Then, $$3 \equiv A(x^2-x+1) + (Bx+C)(x+1)$$
$$\equiv (A+B)x^2 + (B+C-A)x + (A+C).$$

Therefore, $$3 = A + C,\ B + C - A = 0,\ A + B = 0, \qquad (\S\ 425)$$

and $$A = 1,\ B = -1,\ C = 2.$$

Therefore, $$\frac{3}{x^3+1} \equiv \frac{1}{x+1} - \frac{x-2}{x^2-x+1}.$$

(3) Resolve $\dfrac{4x^3-x^2-3x-2}{x^2(x+1)^2}$ into partial fractions.

The denominators will be x, x^2, $x+1$, $(x+1)^2$.

Assume $$\frac{4x^3-x^2-3x-2}{x^2(x+1)^2} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}.$$

$$\therefore 4x^3 - x^2 - 3x - 2 \equiv Ax(x+1)^2 + B(x+1)^2 + Cx^2(x+1) + Dx^2$$
$$\equiv (A+C)x^3 + (2A+B+C+D)x^2 + (A+2B)x + B.$$

Therefore, $$A + C = 4, \qquad (\S\ 425)$$
$$2A + B + C + D = -1,$$
$$A + 2B = -3,$$
$$B = -2;$$

and $$A = 1,\ B = -2,\ C = 3,\ D = -4.$$

Therefore, $$\frac{4x^3-x^2-3x-2}{x^2(x+1)^2} \equiv \frac{1}{x} - \frac{2}{x^2} + \frac{3}{x+1} - \frac{4}{(x+1)^2}.$$

Exercise 61

Resolve into partial fractions:

1. $\dfrac{7x+1}{(x+4)(x-5)}$.
2. $\dfrac{6}{(x+3)(x+4)}$.
3. $\dfrac{5x-1}{(2x-1)(x-5)}$.
4. $\dfrac{x-2}{x^2-3x-10}$.
5. $\dfrac{3}{x^3-1}$.
6. $\dfrac{x^2-x-3}{x(x^2-4)}$.
7. $\dfrac{3x^2-4}{x^2(x+5)}$.
8. $\dfrac{7x^2-x}{(x-1)^2(x+2)}$.
9. $\dfrac{2x^2-7x+1}{x^3-1}$.
10. $\dfrac{7x-1}{6x^2-5x+1}$.
11. $\dfrac{13x+46}{12x^2-11x-15}$.
12. $\dfrac{2x^2-11x+5}{x^3-x^2-11x+15}$.
13. $\dfrac{x^2-15x-18}{(x+3)(x-3)(x-1)}$.
14. $\dfrac{3x^2+12x+11}{(x+1)(x+2)(x+3)}$.

EXPANSION IN SERIES

427. A series which is obtained from a given expression is called the **expansion** of that expression. The given expression is called the **generating function** of the series.

Thus (§ 389), the expression $\dfrac{1}{1-x}$ is the generating function of the infinite series $1+x+x^2+x^3+\cdots$

If the series is finite, the generating function is equal to the expansion for all values of the symbols involved.

Thus, $$\left(\frac{1+2x^2}{x}\right)^3 \equiv \frac{1}{x^3}+\frac{6}{x}+12x+8x^3.$$

If the series is infinite, the generating function is equal to the expansion for only such values of the symbols involved as make the expansion a convergent series.

Thus, $\dfrac{1}{1-x}$ is equal to the series $1+x+x^2+x^3+\cdots$ when, and only when, x is numerically less than 1 (§ 390).

(1) Expand $\frac{x}{1+x^2}$ in ascending powers of x.

Divide x by $1+x^2$; then,

$$\frac{x}{1+x^2} = x - x^3 + x^5 - \cdots$$

provided x is so taken that the series is convergent. By §§ 407, 405, the value of x must be numerically less than 1.

(2) Expand $\frac{x}{1+x^2}$ in descending powers of x.

Divide x by x^2+1; then,

$$\frac{x}{1+x^2} = \frac{1}{x} - \frac{1}{x^3} + \frac{1}{x^5} - \cdots$$

provided x is so taken that the series is convergent. By §§ 407, 405, the value of x must be numerically greater than 1.

In the two preceding examples we have found an expansion of $\frac{x}{1+x^2}$ for all values of x except ± 1.

(3) Expand $\frac{x}{1+x^2}$ in ascending powers of x by the binomial theorem.

$$\frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - \cdots$$

$$\therefore \frac{x}{1+x^2} = x - x^3 + x^5 - \cdots$$

provided x is so taken that the series is convergent.

(4) Expand $\frac{2+3x}{1+x+x^2}$ in ascending powers of x.

Assume $$\frac{2+3x}{1+x+x^2} = A + Bx + Cx^2 + Dx^3 + \cdots$$

Clear of fractions,

$$\begin{aligned} 2+3x = A + Bx &+ Cx^2 + Dx^3 + \cdots \\ + Ax &+ Bx^2 + Cx^3 + \cdots \\ &+ Ax^2 + Bx^3 + \cdots \end{aligned}$$

Hence, $A = 2$, $B + A = 3$, $C + B + A = 0$, $D + C + B = 0$. (§ 425)

Whence, $A = 2, \quad B = 1, \quad C = -3, \quad D = 2$; and so on.

$$\therefore \frac{2+3x}{1+x+x^2} = 2 + x - 3x^2 + 2x^3 + x^4 - 3x^5 + \cdots$$

The series is of course equal to the fraction for only such values of x as make the series convergent.

REMARK. In employing the method of Indeterminate Coefficients the form of the given expression must determine what powers of the variable x must be assumed. It is *necessary* and *sufficient* that the assumed equation, when simplified, shall have in the right member all the powers of x that are found in the left member.

If any powers of x occur in the *right* member that are not in the *left* member, the coefficients of these powers in the right member will vanish, so that in this case the method still applies; but if any powers of x occur in the *left* member that are not in the *right* member, then the coefficients of these powers of x must be put equal to 0 in equating the coefficients of like powers of x; and this leads to absurd results. Thus, if it were assumed in Example (4) that

$$\frac{2+3x}{1+x+x^2} = Ax + Bx^2 + Cx^3 + \cdots,$$

there would be in the simplified equation no term on the right corresponding to 2 on the left; so that, in equating the coefficients of like powers of x, 2, which is $2x^0$, would have to be put equal to $0x^0$; that is, $2 = 0$, an absurdity.

(5) Expand $(a - x)^{\frac{1}{2}}$ in a series of ascending powers of x.

Assume $\quad (a - x)^{\frac{1}{2}} = A + Bx + Cx^2 + Dx^3 + \cdots$

Square, $a - x = A^2 + 2ABx + (2AC + B^2)x^2 + (2AD + 2BC)x^3 + \cdots$

Therefore, by § 425,

$$A^2 = a,\ 2AB = -1,\ 2AC + B^2 = 0,\ 2AD + 2BC = 0, \text{ etc.},$$

and
$$A = a^{\frac{1}{2}},\ B = -\frac{1}{2a^{\frac{1}{2}}},\quad C = -\frac{1}{8a^{\frac{3}{2}}},\quad D = -\frac{1}{16a^{\frac{5}{2}}}.$$

Hence,
$$(a - x)^{\frac{1}{2}} = a^{\frac{1}{2}} - \frac{x}{2a^{\frac{1}{2}}} - \frac{x^2}{8a^{\frac{3}{2}}} - \frac{x^3}{16a^{\frac{5}{2}}} - \cdots$$

(6) Expand $\dfrac{7+x}{(1+x)(1+x^2)}$ in ascending powers of x.

Assume $$\frac{7+x}{(1+x)(1+x^2)} \equiv \frac{A}{1+x} + \frac{Bx+C}{1+x^2}.$$

$$\therefore 7 + x \equiv (A + C) + (B + C)x + (A + B)x^2.$$

$$\therefore A + C = 7, \quad B + C = 1, \quad A + B = 0.$$

Whence, $$A = 3, \qquad B = -3, \qquad C = 4.$$

$$\therefore \frac{7+x}{(1+x)(1+x^2)} \equiv \frac{3}{1+x} + \frac{4-3x}{1+x^2}.$$

Now, $$\frac{4-3x}{1+x^2} = (4-3x)\left(\frac{1}{1+x^2}\right) = (4-3x)(1 - x^2 + x^4 - \cdots)$$
$$= 4 - 3x - 4x^2 + 3x^3 + 4x^4 - \cdots,$$

and $$\frac{3}{1+x} = 3\left(\frac{1}{1+x}\right) = 3 - 3x + 3x^2 - 3x^3 + 3x^4 - \cdots$$

Add, $$\frac{7+x}{(1+x)(1+x^2)} = 7 - 6x - x^2 + 7x^4 - \cdots$$

REVERSION OF A SERIES

428. Reversion of a Series. If y is the sum of a convergent series in x, the writing of x in terms of a convergent series in y is called the *reversion of the series.*

Given $y = ax + bx^2 + cx^3 + dx^4 + \cdots$, where the series is convergent, to find x in terms of a convergent series in y.

Assume $$x = Ay + By^2 + Cy^3 + Dy^4 + \cdots$$

In this series for y put $ax + bx^2 + cx^3 + dx^4 + \cdots$;

$$x = aAx + \begin{array}{r|l|l|l} bA & x^2 + cA & x^3 + dA & x^4 + \cdots \\ + a^2B & \quad + 2\,abB & \quad + b^2B & \\ & \quad + a^3C & \quad + 2\,acB & \\ & & \quad + 3\,a^2bC & \\ & & \quad + a^4D & \end{array}$$

Equate coefficients (§ 425),

$$aA=1;\quad bA+a^2B=0;\quad cA+2abB+a^3C=0;$$

$$dA+b^2B+2acB+3a^2bC+a^4D=0.$$

$$\therefore A=\frac{1}{a},\quad B=-\frac{b}{a^3},\quad C=\frac{2b^2-ac}{a^5},$$

$$D=-\frac{5b^3-5abc+a^2d}{a^7},\text{ etc.}$$

(1) Revert $y=x+x^2+x^3+\cdots$

Here, $a=1,\quad b=1,\quad c=1,\quad d=1,\cdots$

$A=1,\quad B=-1,\quad C=1,\quad D=-1,\cdots$

Hence, $x=y-y^2+y^3-y^4+\cdots$

(2) Revert $y=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\cdots$

Here, $a=1,\quad b=-\frac{1}{2},\quad c=\frac{1}{3},\quad d=-\frac{1}{4},\cdots$

$$\therefore A=1,\quad B=\frac{1}{\lfloor 2},\quad C=\frac{1}{\lfloor 3},\quad D=\frac{1}{\lfloor 4},\cdots$$

Hence, $x=y+\dfrac{y^2}{\lfloor 2}+\dfrac{y^3}{\lfloor 3}+\dfrac{y^4}{\lfloor 4}+\cdots$

Exercise 62

Expand to four terms in ascending powers of x:

1. $\dfrac{1}{1-2x}$.
2. $\dfrac{1}{2-3x}$.
3. $\dfrac{1+x}{2+3x}$.
4. $\dfrac{1-x}{1+x+x^2}$.
5. $\dfrac{5-2x}{1+x-x^2}$.
6. $\dfrac{4x-6x^2}{1-2x+3x^2}$.
7. $\dfrac{x(x-1)}{(x+1)(x^2+1)}$.
8. $\dfrac{x^2-x+1}{x^2(x^2-1)}$.
9. $\dfrac{2x^2-1}{x(x^2+1)}$.

Expand to four terms in descending powers of x:

10. $\dfrac{4}{2+x}$.

11. $\dfrac{2-x}{3+x}$.

12. $\dfrac{5-2x}{1+3x-x^2}$.

13. $\dfrac{x^2-x+1}{x(x-2)}$.

14. $\dfrac{3x-2}{x(x-1)^2}$.

15. $\dfrac{x^2-x+1}{(x-1)(x^2+1)}$.

Revert:

16. $y = x - 2x^2 + 3x^3 - 4x^4 + \cdots$

17. $y = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$

18. $y = x + \dfrac{x^2}{1\cdot 2} + \dfrac{x^3}{2\cdot 3} + \dfrac{x^4}{3\cdot 4} + \cdots$

RECURRING SERIES

429. From the expression $\dfrac{1+x}{1-2x-x^2}$ we obtain by actual division, or by the method of indeterminate coefficients, the infinite series

$$1 + 3x + 7x^2 + 17x^3 + 41x^4 + 99x^5 + \cdots$$

In this series any required term after the second is found by multiplying the term before the required term by $2x$, the term before that by x^2, and adding the products.

Thus, take the fifth term:

$$41x^4 = 2x(17x^3) + x^2(7x^2).$$

In general, if u_n represents the nth term,

$$u_n \equiv 2xu_{n-1} + x^2u_{n-2}.$$

A series in which a relation of this character exists is called a recurring series. Recurring series are of the *first*, *second*, *third*, $\cdots$ order, according as each term is dependent upon *one*, *two*, *three*, $\cdots$ preceding terms.

A recurring series of the first order is evidently an ordinary geometrical series.

In an arithmetical or a geometrical series any required term can be found when the term immediately preceding is given. In a series of differences or in a recurring series several preceding terms must be given if any required term is to be found.

The relation which exists between the successive terms is called the **identical relation** of the series; the coefficients of this relation, when all the terms are transposed to the left member, is called the **scale of relation** of the series.

Thus, in the series

$$1 + 3x + 7x^2 + 17x^3 + 41x^4 + 99x^5 + \cdots$$

the identical relation is

$$u_n \equiv 2xu_{n-1} + x^2u_{n-2};$$

and the scale of relation is $\quad 1 - 2x - x^2.$

430. If the identical relation of the series is given, any required term can be found when a sufficient number of preceding terms is given.

Conversely, the identical relation can be found when a sufficient number of terms is given.

(1) Find the identical relation of the recurring series

$$1 + 4x + 14x^2 + 49x^3 + 171x^4 + 597x^5 + 2084x^6 + \cdots$$

Try first a relation of the second order.

Assume $\quad u_n = pxu_{n-1} + qx^2u_{n-2}.$

Put $n = 3$, and then $n = 4$.

$$14 = 4p + q,$$
$$49 = 14p + 4q;$$

whence, $\quad p = \frac{7}{2},\ q = 0.$

This gives a relation which does not hold true for the fifth and following terms.

Try next a relation of the third order.

Assume $\quad u^n = pxu_{n-1} + qx^2u_{n-2} + rx^3u_{n-3}.$

Put $n = 4$, then $n = 5$, then $n = 6$.

$$49 = 14p + 4q + r,$$
$$171 = 49p + 14q + 4r,$$
$$597 = 171p + 49q + 14r;$$

whence, $p = 3,\ q = 2,\ r = -1.$

This gives the relation

$$u_n \equiv 3xu_{n-1} + 2x^2u_{n-2} - x^3u_{n-3},$$

which is found to hold true for the seventh term.

The *scale of relation* is $1 - 3x - 2x^2 + x^3$.

(2) Find the eighth term of the above series.

Here,
$$\begin{aligned} u_8 &\equiv 3xu_7 + 2x^2u_6 - x^3u_5 \\ &\equiv 3x(2084x^6) + 2x^2(597x^5) - x^3(171x^4) \\ &\equiv 7275x^7. \end{aligned}$$

SUMMATION OF SERIES

431. Infinite Series. By the *sum* of an infinite convergent *numerical* series is meant the limit which the sum of n terms of the series approaches as n is indefinitely increased. A non-convergent numerical series has no true sum.

By the sum of an infinite series of which the successive terms involve one or more *variables* is meant the *generating function* of the series (§ 427), that is, *the expression of which the series is the expansion.*

The generating function is a true sum when, and only when, the series is convergent.

The process of finding the generating function is called *summation* of the series.

432. Recurring Series. The sum of a recurring series can be found by a method analogous to that by which the sum of a geometrical series is found (§ 276).

Take, for example, a recurring series of the second order in which the identical relation is

$$u_k \equiv pu_{k-1} + qu_{k-2},$$

or
$$u_k - pu_{k-1} - qu_{k-2} \equiv 0.$$

Let s represent the sum of the series; then,

$$\begin{aligned} s &= u_1 + u_2 + u_3 + \cdots + u_{n-1} + u_n,\\ -ps &= \quad -pu_1 - pu_2 - \cdots - pu_{n-2} - pu_{n-1} - pu_n,\\ -qs &= \qquad -qu_1 - \cdots - qu_{n-3} - qu_{n-2} - qu_{n-1} - qu_n. \end{aligned}$$

Now, by the identical relation,

$$u_3 - pu_2 - qu_1 = 0,\ u_4 - pu_3 - qu_2 = 0,\ \cdots,\ u_n - pu_{n-1} - qu_{n-2} = 0.$$

Therefore, adding the above series,

$$s = \frac{u_1 + (u_2 - pu_1)}{1 - p - q} - \frac{pu_n + q(u_n + u_{n-1})}{1 - p - q}.$$

Observe that the denominator is the *scale of relation*.

If the series is infinite and convergent, u_n and u_{n-1} each approaches 0 as a limit, and s approaches as a limit the fraction $\dfrac{u_1 + (u_2 - pu_1)}{1 - p - q}$.

If the series is infinite, whether convergent or not, this fraction is the *generating function* of the series.

For a recurring series of the third order of which the identical relation is

$$u_k \equiv pu_{k-1} + qu_{k-2} + ru_{k-3},$$

we find
$$s = \frac{u_1 + (u_2 - pu_1) + (u_3 - pu_2 - qu_1)}{1 - p - q - r}$$
$$- \frac{pu_n + q(u_n + u_{n-1}) + r(u_n + u_{n-1} + u_{n-2})}{1 - p - q - r}.$$

Similarly for a recurring series of higher order.

(1) Find the generating function of the infinite recurring series

$$1+4x+13x^2+43x^3+142x^4+\cdots$$

By § 430 the identical relation is found to be

$$u_k \equiv 3xu_{k-1}+x^2u_{k-2}.$$

Hence,
$$\begin{aligned} s &= 1+4x+13x^2+43x^3+142x^4+\cdots \\ -3xs &= \quad\;\, -3x-12x^2-39x^3-129x^4-\cdots \\ -x^2s &= \qquad\qquad\;\; -x^2-\;4x^3-\;13x^4-\cdots \end{aligned}$$

Add,
$$(1-3x-x^2)s=1+x.$$

$$\therefore s=\frac{1+x}{1-3x-x^2}.$$

(2) Find the generating function and the general term of the infinite recurring series

$$1-7x-x^2-43x^3-49x^4-307x^5-\cdots$$

Here,
$$u_k \equiv xu_{k-1}+6x^2u_{k-2}.$$

$$\begin{aligned} s &= 1-7x-\;x^2-43x^3-49x^4-\cdots \\ -\;xs &= \quad\;\, -\;x+7x^2+\;x^3+43x^4+\cdots \\ -6x^2s &= \qquad\qquad -6x^2+42x^3+\;6x^4+\cdots \end{aligned}$$

Add,
$$(1-x-6x^2)s=1-8x.$$

$$\therefore s=\frac{1-8x}{1-x-6x^2}=\frac{1-8x}{(1+2x)(1-3x)}.$$

By § 426 we find

$$\frac{1-8x}{(1+2x)(1-3x)}=\frac{2}{1+2x}-\frac{1}{1-3x}.$$

By the binomial theorem or by actual division,

$$\frac{1}{1+2x}=1-2x+2^2x^2-2^3x^3+\cdots+2^r(-1)^rx^r+\cdots,$$

$$\frac{1}{1-3x}=1+3x+3^2x^2+3^3x^3+\cdots+3^rx^r+\cdots$$

Hence, the general term of the given series is

$$[2^{r+1}(-1)^r-3^r]x^r.$$

(3) Find the identical relation in the series

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + \cdots$$

The identical relation is found from the equations

$$\begin{aligned} 16 &= 9p + 4q + r, \\ 25 &= 16p + 9q + 4r, \\ 36 &= 25p + 16q + 9r, \end{aligned}$$

to be $u_k \equiv 3u_{k-1} - 3u_{k-2} + u_{k-3}$.

Exercise 63

Find the identical relation and the generating function of:

1. $1 + 2x + 7x^2 + 23x^3 + 76x^4 + \cdots$
2. $3 + 2x + 3x^2 + 7x^3 + 18x^4 + \cdots$
3. $3 + 5x + 9x^2 + 15x^3 + 23x^4 + 33x^5 + 45x^6 + \cdots$
4. $1 + 4x + 11x^2 + 27x^3 + 65x^4 + 158x^5 + 388x^6 + \cdots$

Find the generating function and the general term of:

5. $2 + 3x + 5x^2 + 9x^3 + 17x^4 + 33x^5 + \cdots$
6. $7 - 6x + 9x^2 + 27x^3 + 54x^4 + 189x^5 + 567x^6 + \cdots$
7. $1 + 5x + 9x^2 + 13x^3 + 17x^4 + 21x^5 + \cdots$
8. $1 + x - 7x^3 + 33x^4 - 130x^5 + 499x^6 - \cdots$
9. $3 + 6x + 14x^2 + 36x^3 + 98x^4 + 276x^5 + 794x^6 + \cdots$
10. $1 + 4x + 9x^2 + 16x^3 + 25x^4 + 36x^5 + 49x^6 + \cdots$

Find the sum of n terms of:

11. $2 + 5 + 10 + 17 + 26 + 37 + 50 + \cdots$
12. $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + \cdots$
13. $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$
14. $1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + 28x^6 + \cdots$
15. $1^2 + 3^2 + 5^2 + 7^2 + 9^2 + 11^2 + \cdots$
16. $1^2 + 5^2 + 9^2 + 13^2 + 17^2 + 21^2 + \cdots$

INTERPOLATION

433. As the expansion of $(a+b)^n$ by the binomial theorem has the same form for fractional as for integral values of n, the formula (§ 421)

$$a_{n+1}=a+nb+\frac{n(n-1)}{1\times 2}c+\frac{n(n-1)(n-2)}{1\times 2\times 3}d+\cdots$$

may be extended to cases in which n is a fraction, and be employed to insert or *interpolate* terms in a series between given terms.

(1) The cube roots of 27, 28, 29, 30 are 3, 3.03659, 3.07232, 3.10723. Find the cube root of 27.9.

	3.00000		3.03659		3.07232		3.10723
First differences,		0.03659		0.03573		0.03491	
Second differences,			− 0.00086		− 0.00082		
Third differences,				0.00004			

These values substituted in the general formula give

$$\begin{aligned}3+\frac{9}{10}(0.03659)+\frac{9}{10}\left(-\frac{1}{10}\right)\left(-\frac{0.00086}{2}\right)+\frac{9}{10}\left(-\frac{1}{10}\right)\left(-\frac{11}{10}\right)\left(\frac{0.00004}{6}\right)\\ =3+0.032931+0.0000387+0.00000066\\ =3.03297.\end{aligned}$$

(2) Given log 127 = 2.1038, log 128 = 2.1072, log 129 = 2.1106. Find log 127.37.

	2.1038		2.1072		2.1106
First differences,		0.0034		0.0034	
Second differences,			0		

Therefore, the differences of the second order vanish, and the required logarithm is

$$\begin{aligned}2.1038+\tfrac{37}{100}\text{ of }0.0034&=2.1038+0.001258\\&=2.1051.\end{aligned}$$

(3) The latitude of the moon on a certain Monday at noon was 1° 53′ 18.9″, at midnight 2° 27′ 8.6″; on Tuesday at noon 2° 58′ 55.2″, at midnight 3° 28′ 5.8″; on Wednesday at noon 3° 54′ 8.8″. Find its latitude at 9 P.M. on Monday.

The series expressed in seconds and the successive differences are

$$\begin{array}{ccccccccc}
6798.9 & & 8828.6 & & 10735.2 & & 12485.8 & & 14048.8 \\
& 2029.7 & & 1906.6 & & 1750.6 & & 1563.0 & \\
& & -123.1 & & -156.0 & & -187.6 & & \\
& & & -32.9 & & -31.6 & & & \\
& & & & 1.3 & & & &
\end{array}$$

As 9 hours $= \frac{3}{4}$ of 12 hours, $n = \frac{3}{4}$.
Also, $a = 6798.9$, $b = 2029.7$, $c = -123.1$, $d = -32.9$, $e = 1.3$.

These values substituted in the general formula

$$a + nb + \frac{n(n-1)}{1 \times 2}c + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}d + \frac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4}e + \cdots$$

give

$$6798.9 + \frac{3}{4}(2029.7) + \frac{3}{4}\left(-\frac{1}{4}\right)\left(-\frac{123.1}{2}\right) + \frac{3}{4}\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{32.9}{6}\right)$$
$$+ \frac{3}{4}\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)\left(\frac{1.3}{24}\right) + \cdots$$
$$= 6798.9 + 1522.28 + 11.54 - 1.29 - 0.03 \cdots$$
$$= 8331.4$$
$$= 2° 18' 51.4''.$$

EXPONENTIAL AND LOGARITHMIC SERIES

434. Exponential Series. By the binomial theorem,

$$\left(1 + \frac{1}{n}\right)^{nx} = 1 + nx \times \frac{1}{n} + \frac{nx(nx-1)}{1 \times 2} \times \frac{1}{n^2}$$
$$+ \frac{nx(nx-1)(nx-2)}{1 \times 2 \times 3} \times \frac{1}{n^3} + \cdots$$
$$= 1 + x + \frac{x\left(x - \frac{1}{n}\right)}{\underline{|2}} + \frac{x\left(x - \frac{1}{n}\right)\left(x - \frac{2}{n}\right)}{\underline{|3}} + \cdots \quad [1]$$

This equation is true for all real values of x. It is, however, true only for values of n numerically greater than 1, since $\frac{1}{n}$ must be numerically less than 1 (§ 418).

As [1] is true for all values of x, it is true when $x = 1$.

$$\therefore \left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1-\frac{1}{n}}{\lfloor 2} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3} + \cdots \qquad [2]$$

But
$$\left[\left(1+\frac{1}{n}\right)^n\right]^x = \left(1+\frac{1}{n}\right)^{nx}. \qquad (§ 299)$$

Hence, from [1] and [2],

$$\left[1 + 1 + \frac{1-\frac{1}{n}}{\lfloor 2} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3} + \cdots\right]^x$$
$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3} + \cdots$$

This last equation is true for all values of n numerically greater than 1. Take the limits of the two members as n increases without limit. Then (§ 383),

$$\left(1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots\right)^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots, \qquad [3]$$

and this is true for all values of x. It is easily seen by § 405 that each series is convergent for all values of x.

The sum of the infinite series in parenthesis is called the *natural base* (§ 302), and is generally represented by e; hence, by [3],

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots \qquad [A]$$

To calculate the value of e, we proceed as follows:

$$\begin{array}{r|l}
 & 1.000000 \\
2 & 1.000000 \\
3 & 0.500000 \\
4 & 0.166667 \\
5 & 0.041667 \\
6 & 0.008333 \\
7 & 0.001388 \\
8 & 0.000198 \\
9 & 0.000025 \\
 & 0.000003
\end{array}$$

Add, $\qquad e = 2.71828.$

To ten places, $\qquad e = 2.7182818284.$

435. In [A] put cx in place of x; then,

$$e^{cx} = 1 + cx + \frac{c^2x^2}{|\underline{2}} + \frac{c^3x^3}{|\underline{3}} + \cdots$$

Put $e^c = a$; then $c = \log_e a$, and $e^{cx} = a^x$.

$$\therefore\ a^x = 1 + x\log_e a + \frac{x^2(\log_e a)^2}{|\underline{2}} + \frac{x^3(\log_e a)^3}{|\underline{3}} + \cdots \qquad [B]$$

Series [B] is known as the **exponential series**.

Series [B] reduces to [A] when we put e for a.

436. Logarithmic Series. In [A] put $e^x = 1 + y$; then,

$$x = \log_e(1 + y), \text{ and by [A]},$$

$$y = x + \frac{x^2}{|\underline{2}} + \frac{x^3}{|\underline{3}} + \frac{x^4}{|\underline{4}} + \cdots$$

Revert the series (§ 428), and we obtain

$$x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots$$

But $\qquad x = \log_e(1 + y).$

$$\therefore\ \log_e(1 + y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots \qquad [C]$$

Similarly from [B],

$$\log_a(1+y)=\frac{1}{\log_e a}\left(y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\cdots\right). \qquad [D]$$

The series in [D] is known as the **logarithmic series**; [D] reduces to [C] when we put e for a.

In [C] and [D], in order to have the series convergent, the value of y must *lie between* -1 and $+1$, or *be equal to* $+1$, by § 409, Example (1).

437. Modulus. Comparing [C] and [D], we obtain

$$\log_a(1+y)=\frac{1}{\log_e a}\log_e(1+y);$$

or, putting N for $1+y$,

$$\log_a N=\frac{1}{\log_e a}\log_e N.$$

Hence, to change logarithms from the base e to the base a, multiply by $\dfrac{1}{\log_e a}=\log_a e$; and conversely (§ 318).

The number by which *natural logarithms* must be multiplied to obtain logarithms to the base a is called the **modulus** of the system of logarithms of which a is the base.

Thus, the modulus of the common system is $\log_{10}e$ (§ 320).

438. Calculation of Logarithms. Since the series in [C] and [D] are not convergent when x is numerically greater than 1, they are not adapted to the calculation of logarithms in general. We obtain a convenient series as follows:

The equation

$$\log_e(1+y)=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\cdots \qquad [1]$$

holds true for all values of y numerically less than 1; therefore, if it holds true for any particular value of y, it will hold true when we put $-y$ for y; this gives

$$\log_e(1-y)=-y-\frac{y^2}{2}-\frac{y^3}{3}-\frac{y^4}{4}-\cdots \qquad [2]$$

Subtract [2] from [1]. Then, since

$$\log_e(1+y)-\log_e(1-y)=\log_e\left(\frac{1+y}{1-y}\right),$$

$$\log_e\left(\frac{1+y}{1-y}\right)=2\left(y+\frac{y^3}{3}+\frac{y^5}{5}+\cdots\right).$$

Put $y=\dfrac{1}{2z+1}$; then $\dfrac{1+y}{1-y}=\dfrac{z+1}{z}$,

and $\log_e\left(\dfrac{z+1}{z}\right)=\log_e(z+1)-\log_e z$

$$=2\left(\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+\cdots\right).\quad [E]$$

This series is convergent for all positive values of z.

Logarithms to any base a can be calculated by the corresponding series obtained from [D]; *viz.*,

$\log_a(z+1)-\log_a z$

$$=\frac{2}{\log_e a}\left(\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+\cdots\right).\quad [F]$$

(1) Calculate to six places of decimals $\log_e 2$, $\log_e 3$, $\log_e 10$, $\log_{10} e$.

In [E] put $z=1$; then $2z+1=3$, $\log_e z=0$,

and $$\log_e 2=\frac{2}{3}+\frac{2}{3\times 3^3}+\frac{2}{5\times 3^5}+\frac{2}{7\times 3^7}+\cdots$$

The work may be arranged as follows:

$$\begin{array}{r|l}
3 & 2.0000000 \\
9 & 0.6666667 \div 1 = 0.6666667 \\
9 & 0.0740741 \div 3 = 0.0246914 \\
9 & 0.0082305 \div 5 = 0.0016461 \\
9 & 0.0009145 \div 7 = 0.0001306 \\
9 & 0.0001016 \div 9 = 0.0000113 \\
9 & 0.0000113 \div 11 = 0.0000010 \\
 & 0.0000013 \div 13 = \underline{0.0000001} \\
 & \qquad \log_e 2 = 0.693147
\end{array}$$

$$\log_e 3 = \log_e 2 + \frac{2}{5} + \frac{2}{3 \times 5^3} + \frac{2}{5 \times 5^5} + \cdots$$

$$= 1.0986123.$$

$$\log_e 9 = \log_e (3^2) = 2 \log_e 3 = 2.1972246.$$

$$\log_e 10 = \log_e 9 + \frac{2}{19} + \frac{2}{3 \times 19^3} + \frac{2}{5 \times 19^5} + \cdots$$

$$= 2.1972246 + 0.1053606$$

$$= 2.302585.$$

$$\log_{10} e = \frac{1}{\log_e 10} = 0.434294.$$

Hence, the *modulus* of the common system is 0.434294 (§ 320).

To *ten* places of decimals:

$$\log_e 10 = 2.3025850928,$$

$$\log_{10} e = 0.4342944819.$$

For calculating common logarithms we use the series in [F]

$$\log_{10}(z+1) - \log_{10} z$$

$$= 0.8685889638\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \cdots\right).$$

(2) Calculate to five places of decimals $\log_{10} 11$.

Put $z = 10$; then $2z + 1 = 21$, $\log z = 1$,

$$\log 11 = 1 + 0.868588\left(\frac{1}{21} + \frac{1}{3 \times 21^3} + \frac{1}{5 \times 21^5} + \cdots\right).$$

$$\begin{array}{r|l}
21 & \underline{0.868588} \\
441 & \underline{0.041361} \div 1 = 0.041361 \\
 & 0.000094 \div 3 = \underline{0.000031} \\
 & \qquad\qquad\quad 0.041392 \\
 & \qquad\qquad\quad \underline{1.\qquad\quad} \\
\multicolumn{2}{r}{\log_{10} 11 = 1.04139}
\end{array}$$

In calculating logarithms the accuracy of the work may be tested every time we come to a composite number by adding the logarithms of the several factors (§ 300). In fact, the logarithms of composite numbers may be found by addition, and then only the logarithms of prime numbers need be found by the series.

439. Limit of $\left(1+\frac{x}{n}\right)^n$. By the binomial theorem,

$$\left(1+\frac{x}{n}\right)^n = 1 + n \times \frac{x}{n} + \frac{n(n-1)}{1 \times 2} \times \frac{x^2}{n^2} + \frac{n(n-1)(n-2)}{1 \times 2 \times 3} \times \frac{x^3}{n^3} + \cdots$$

$$= 1 + x + \frac{1-\frac{1}{n}}{\lfloor 2}x^2 + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3}x^3 + \cdots$$

This equation is true for all values of n greater than x (§ 418). Take the limit as n increases without limit, x remaining finite.

Then, $$\lim_{n \doteq \infty}\left(1+\frac{x}{n}\right)^n = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$$
$$= e^x$$
$$= \lim_{n \doteq \infty}\left(1+\frac{1}{n}\right)^{nx}. \qquad (\S\ 434)$$

Exercise 64

1. Show that the infinite series

$$\frac{1}{1 \times 2} - \frac{1}{2 \times 2^2} + \frac{1}{3 \times 2^3} - \frac{1}{4 \times 2^4} + \cdots$$

is convergent, and find its sum.

2. Find the limit which $\sqrt[n]{1+nx}$ approaches as n approaches 0 as a limit.

3. Show that $\frac{1}{e} = 2\left(\frac{1}{\lfloor 3} + \frac{2}{\lfloor 5} + \frac{3}{\lfloor 7} + \cdots\right)$.

4. Calculate to four places, $\log_e 4$, $\log_e 5$, $\log_e 6$, $\log_e 7$.

5. Find to four places the moduli of the systems of which the bases are 2, 3, 4, 5, 6, 7.

6. Show that

$$\log_e\left(\frac{8}{e}\right)=\frac{5}{1\times 2\times 3}+\frac{7}{3\times 4\times 5}+\frac{9}{5\times 6\times 7}+\cdots$$

7. Show that

$$\log_e a-\log_e b=\frac{a-b}{a}+\frac{1}{2}\left(\frac{a-b}{a}\right)^2+\frac{1}{3}\left(\frac{a-b}{a}\right)^3+\cdots$$

8. Show that, if x is positive,

$$x+\frac{1}{x}-\frac{1}{2}\left(x^2+\frac{1}{x^2}\right)+\frac{1}{3}\left(x^3+\frac{1}{x^3}\right)-\cdots=\log_e\left(2+x+\frac{1}{x}\right).$$

9. Show that $1+\dfrac{2^3}{\lfloor 2}+\dfrac{3^3}{\lfloor 3}+\dfrac{4^3}{\lfloor 4}+\cdots=5e.$

10. Show that $e^{x\sqrt{-1}}=X+Y\sqrt{-1}$, where

$$X=1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}-\frac{x^6}{\lfloor 6}+\cdots,\quad Y=x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}-\frac{x^7}{\lfloor 7}+\cdots$$

11. Expand $\dfrac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2}$ in ascending powers of x.

12. Expand $\dfrac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{2\sqrt{-1}}$ in ascending powers of x.

13. Find the sum of n terms of the series

$$\frac{a}{b}+\frac{a(a+x)}{b(b+x)}+\frac{a(a+x)(a+2x)}{b(b+x)(b+2x)}+\cdots$$

14. Show that, if n is any positive integer,

$$\frac{n}{n+1}-\frac{n(n-1)}{(n+1)(n+2)}+\frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)}-\cdots$$
$$\pm\frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n+1)(n+2)\cdots(2n-1)(2n)}=\frac{1}{2}.$$

CHAPTER XXVI

CONTINUED FRACTIONS

440. A fraction in the form

$$\cfrac{a}{b+\cfrac{c}{d+\cfrac{e}{f+\text{etc.}}}}$$

is called a **continued fraction.**

A continued fraction in which each of the numerators of the component fractions is $+1$ and each of the denominators is a positive integer, as

$$\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+\text{etc.}}}},$$

is called a **simple continued fraction.**

We shall consider in this chapter some of the elementary properties of simple continued fractions.

441. *Any proper fraction in its lowest terms may be converted into a terminated simple continued fraction.*

Let $\frac{b}{a}$ be a fraction in its lowest terms.

Then, if p is the quotient and c the remainder of $a \div b$,

$$\frac{b}{a}=\frac{1}{\frac{a}{b}}=\frac{1}{p+\frac{c}{b}}.$$

If q is the quotient and d the remainder of $b \div c$,

$$\frac{1}{p+\frac{c}{b}} = \frac{1}{p+\frac{1}{\frac{b}{c}}} = \frac{1}{p+\frac{1}{q+\frac{d}{c}}}.$$

Hence,

$$\frac{a}{b} = \frac{1}{p+\frac{1}{q+\frac{1}{r+\text{etc.}}}}$$

The successive steps of the process are the same as the steps in finding the H.C.F. of a and b; and since a and b are prime to each other, a remainder 1 will at length be reached, and the fraction terminates.

Observe that $p, q, r, \cdots$ are all positive integers.

442. Convergents. The fractions formed by taking one, two, three, $\cdots$ of the quotients $p, q, r, \cdots$ are

$$\frac{1}{p}, \quad \frac{1}{p+\frac{1}{q}}, \quad \frac{1}{p+\frac{1}{q+\frac{1}{r}}}, \cdots,$$

which simplified are

$$\frac{1}{p}, \quad \frac{q}{pq+1}, \quad \frac{qr+1}{(pq+1)r+p}, \cdots$$

and are called the first, *second, third,* $\cdots$ *convergents* respectively.

The value of the complete continued fraction is called briefly the **complete value**.

443. *The successive convergents are alternately greater than and less than the complete value of the continued fraction.*

Let x be the complete value of

$$\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+\text{etc.}}}}$$

Then, since $p, q, r, \cdots$ are positive integers,

$$p < p + \cfrac{1}{q+\cfrac{1}{r+\text{etc.}}}$$

$$\therefore \frac{1}{p} > \cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+\text{etc.}}}}$$

That is, $$\frac{1}{p} > x.$$

Again, $$q < q + \frac{1}{r+\text{etc.}}$$

$$\therefore \frac{1}{q} > \cfrac{1}{q+\cfrac{1}{r+\text{etc.}}}$$

$$\therefore \cfrac{1}{p+\cfrac{1}{q}} < \cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+\text{etc.}}}}$$

That is, $$\cfrac{1}{p+\cfrac{1}{q}} < x; \text{ and so on.}$$

Corollary. Hence, if $\frac{u_1}{v_1}$, $\frac{u_2}{v_2}$ are consecutive convergents to x, then $x >$ or $< \frac{u_2}{v_2}$ according as $\frac{u_1}{v_1} >$ or $< \frac{u_2}{v_2}$, and, therefore, $x^2 >$ or $< \frac{u_2^2}{v_2^2}$ according as $\frac{u_1}{v_1} >$ or $< \frac{u_2}{v_2}$.

Therefore, $v_2^2x^2 - u_2^2$ and $u_1v_2 - u_2v_1$ are simultaneously positive or simultaneously negative. Therefore, $\dfrac{v_2^2x^2 - u_2^2}{u_1v_2 - u_2v_1}$ *is always positive.*

Note. Continued fractions are often written in a compact and convenient form; thus, the fraction

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e}}}}$$

may be written in the form $a + \dfrac{1}{b+}\,\dfrac{1}{c+}\,\dfrac{1}{d+}\,\dfrac{1}{e}.$

444. *If* $\dfrac{u_1}{v_1}, \dfrac{u_2}{v_2}, \dfrac{u_3}{v_3}$ *are any three consecutive convergents, and if* m_1, m_2, m_3 *are the quotients that produced them, then*

$$\frac{u_3}{v_3} = \frac{m_3u_2 + u_1}{m_3v_2 + v_1}.$$

For, if the first three quotients are p, q, r, the first three convergents are (§ 442)

$$\frac{1}{p}, \quad \frac{q}{pq+1}, \quad \frac{qr+1}{(pq+1)r+p}. \qquad [1]$$

From § 442 it is seen that the second convergent is formed from the first by writing in it $p + \dfrac{1}{q}$ for p; and the third from the second by writing $q + \dfrac{1}{r}$ for q. In this way any convergent may be formed from the preceding convergent.

Therefore, $\dfrac{u_3}{v_3}$ is formed from $\dfrac{u_2}{v_2}$ by writing $m_2 + \dfrac{1}{m_3}$ for m_2.

The numerator of the third convergent in [1] is equal to

$r \times$ (second numerator) + (first numerator).

The denominator of the third convergent in [1] is equal to

$r \times$ (second denominator) + (first denominator).

Assume that this law holds true for the third of the three consecutive convergents

$$\frac{u_0}{v_0},\quad \frac{u_1}{v_1},\quad \frac{u_2}{v_2},$$

so that
$$\frac{u_2}{v_2}=\frac{m_2u_1+u_0}{m_2v_1+v_0}. \qquad [2]$$

Then, since $\frac{u_3}{v_3}$ is formed from $\frac{u_2}{v_2}$ by using $m_2+\frac{1}{m_3}$ for m_2,

$$\frac{u_3}{v_3}=\frac{\left(m_2+\frac{1}{m_3}\right)u_1+u_0}{\left(m_2+\frac{1}{m_3}\right)v_1+v_0}=\frac{m_3(m_2u_1+u_0)+u_1}{m_3(m_2v_1+v_0)+v_1}.$$

Substitute u_2 and v_2 for their values $m_2u_1+u_0$ and $m_2v_1+v_0$.

Then,
$$\frac{u_3}{v_3}=\frac{m_3u_2+u_1}{m_3v_2+v_1}.$$

Therefore, the law still holds true; and as it has been shown to be true for the third convergent, the law is general by mathematical induction.

Corollary. If $\frac{m_1}{1}, \frac{u_2}{v_2}, \frac{u_3}{v_3}, \dots, \frac{u_n}{v_n}, \cdots$ are the convergents to $m_1+\frac{1}{m_2+}\frac{1}{m_3+}\cdots\frac{1}{+m_n+}\cdots$, then, since $u_n=m_nu_{n-1}+u_{n-2}$,

$$\frac{u^n}{u_{n-1}}=m_n+\frac{u_{n-2}}{u_{n-1}}=m_n+\frac{1}{\frac{u_{n-1}}{u_{n-2}}}$$

$$=m_n+\frac{1}{m_{n-1}+}\frac{1}{\frac{u_{n-2}}{u_{n-3}}}$$

$$=m_n+\frac{1}{m_{n-1}+}\frac{1}{m_{n-2}+}\frac{1}{\frac{u_{n-3}}{u_{n-4}}},$$

and so on, until finally

$$\frac{u_n}{u_{n-1}} = m_n + \frac{1}{m_{n-1}+}\,\frac{1}{m_{n-2}+}\cdots\frac{1}{+m_2}\,\frac{1}{+m_1}.$$

In like manner, it may be shown that

$$\frac{v_n}{v_{n-1}} = m_n + \frac{1}{m_{n-1}+}\,\frac{1}{m_{n-2}+}\cdots\frac{1}{+m_2}.$$

445. Examples. (1) Find the continued fraction equal to $\frac{31}{75}$, and also the successive convergents.

Following the process of finding the H.C.F. of 31 and 75, the successive quotients are found to be 2, 2, 2, 1, 1, 2. Hence, the equivalent continued fraction is

$$\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}}}},$$

or, in the compact form,

$$\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2}.$$

To find the successive convergents, write the successive quotients in order in a line, and in the next line below write the initial convergents $\frac{1}{0}$ and $\frac{0}{1}$ to the left of the first quotient; then, beginning with these initial convergents, form the successive convergents as follows: Multiply the $\begin{Bmatrix}\text{numerator}\\ \text{denominator}\end{Bmatrix}$ of any known convergent by the quotient next on its right and to the product add the $\begin{Bmatrix}\text{numerator}\\ \text{denominator}\end{Bmatrix}$ of the convergent next preceding. The sum is the $\begin{Bmatrix}\text{numerator}\\ \text{denominator}\end{Bmatrix}$ of the next succeeding convergent (§ 444). Write this convergent immediately below the quotient producing it.

Thus, Quotients = 2, 2, 2, 1, 1, 2.

Convergents = $\frac{1}{0}$, $\frac{0}{1}$, $\frac{1}{2}$, $\frac{2}{5}$, $\frac{5}{12}$, $\frac{7}{17}$, $\frac{12}{29}$, $\frac{31}{75}$.

If the given fraction is improper with an integral part n, the initial convergents are $\frac{1}{0}$ and $\frac{n}{1}$. Thus, the zeroth convergent is always $\frac{1}{0}$ and the first convergent is the integral part of the continued fraction, or is zero if there is no integral part.

(2) Find the successive convergents to the continued fraction

$$1+\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{5}.$$

$$\text{Quotients} = \quad 2,\ 3,\ 4,\ 5.$$
$$\text{Convergents} = \tfrac{1}{0},\ \tfrac{1}{1},\ \tfrac{3}{2},\ \tfrac{10}{7},\ \tfrac{43}{30},\ \tfrac{225}{157}.$$

Exercise 65

Express the following numbers as simple continued fractions and find the successive convergents:

1. $\frac{20}{29}$.	4. $\frac{1363}{740}$.	7. 0.0498756.
2. $\frac{117}{326}$.	5. $\frac{1240085}{376813}$.	8. 1.4142.
3. $\frac{252}{365}$.	6. 0.43589.	9. 2.44949.

10. Find the value of

$$\frac{1}{4+}\,\frac{1}{3+}\,\frac{1}{2};\quad \frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{7};\quad \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{5}.$$

11. Find a series of fractions converging to the continued fraction that has as quotients 2, 1, 3, 1, 7, 2, 1, 2, 6, 4.

446. *The difference between two consecutive convergents* $\frac{u_1}{v_1}$ *and* $\frac{u_2}{v_2}$ *is* $\frac{1}{v_1 v_2}$.

The difference between the first two convergents is

$$\frac{1}{p}-\frac{q}{pq+1}=\frac{1}{p\,(pq+1)}.$$

Let the sign $\sim$ stand for the words *the difference between*, and assume the proposition true for $\frac{u_0}{v_0}$ and $\frac{u_1}{v_1}$.

Then, $$\frac{u_0}{v_0} \sim \frac{u_1}{v_1} = \frac{u_0v_1 \sim u_1v_0}{v_0v_1} = \frac{1}{v_0v_1}.$$

But

$$\frac{u_2}{v_2} \sim \frac{u_1}{v_1} = \frac{u_2v_1 \sim u_1v_2}{v_1v_2} = \frac{(m_2u_1 + u_0)\,v_1 \sim u_1\,(m_2v_1 + v_0)}{v_1v_2},$$

if we put for u_2 and v_2 their values, $m_2u_1 + u_0$ and $m_2v_1 + v_0$.

Reduce, $$\frac{u_2}{v_2} \sim \frac{u_1}{v_1} = \frac{u_0v_1 \sim u_1v_0}{v_1v_2} = \frac{1}{v_1v_2} \text{ (by assumption).}$$

Hence, if the proposition is true for one pair of consecutive convergents, it is true for the next pair; but it has been shown to be true for the *first* pair; therefore, it is true for *every* pair by mathematical induction.

Corollary. If $\frac{u_1}{v_1}$ and $\frac{u_2}{v_2}$ are two consecutive convergents, $u_1v_2 - u_2v_1 = +1$ or -1 according as $\frac{u_1}{v_1} >$ or $< \frac{u_2}{v_2}$.

447. Since by § 443 the complete value of x lies between two consecutive convergents $\frac{u_1}{v_1}$ and $\frac{u_2}{v_2}$, the convergent $\frac{u_1}{v_1}$ differs from x by a number less than $\frac{u_1}{v_1} \sim \frac{u_2}{v_2}$, that is, by a number less than $\frac{1}{v_1v_2}$; so that the error in taking $\frac{u_1}{v_1}$ for x is less than $\frac{1}{v_1v_2}$, and therefore less than $\frac{1}{m_2v_1^2}$, as $v_2 > m_2v_1$, since $v_2 = m_2v_1 + v_0$.

Hence, the best convergents to select are those immediately preceding large quotients.

448. Any convergent $\frac{u_1}{v_1}$ is in its lowest terms; for, if u_1 and v_1 had any common factor, it would also be a factor of $u_1v_2 \sim u_2v_1$, that is, a factor of 1.

449. *The successive convergents approach more and more nearly to the complete value of the continued fraction.*

Let $\frac{u_0}{v_0}, \frac{u_1}{v_1}, \frac{u_2}{v_2}$ be consecutive convergents.

Now, $\frac{u_2}{v_2}$ differs from x, the value of the complete fraction, only because m_2 is used instead of $m_2 + \frac{1}{m_3 + \text{etc.}}$.

Let this complete quotient, which is always greater than unity, be represented by M.

Then, since $\frac{u_2}{v_2} = \frac{m_2 u_1 + u_0}{m_2 v_1 + v_0}$, $x = \frac{Mu_1 + u_0}{Mv_1 + v_0}$.

$$\therefore x \sim \frac{u_1}{v_1} = \frac{Mu_1 + u_0}{Mv_1 + v_0} \sim \frac{u_1}{v_1} = \frac{u_0 v_1 \sim u_1 v_0}{v_1(Mv_1 + v_0)} = \frac{1}{v_1(Mv_1 + v_0)},$$

and $\frac{u_0}{v_0} \sim x = \frac{u_0}{v_0} \sim \frac{Mu_1 + u_0}{Mv_1 + v_0} = \frac{M(u_0 v_1 \sim u_1 v_0)}{v_0(Mv_1 + v_0)} = \frac{M}{v_0(Mv_1 + v_0)}$.

Now, $1 < M$ and $v_1 > v_0$, and for both these reasons

$$x \sim \frac{u_1}{v_1} < \frac{u_0}{v_0} \sim x.$$

That is, $\frac{u_1}{v_1}$ is nearer to x than is $\frac{u_0}{v_0}$.

Corollary. Hence, the *odd*-numbered convergents to the continued fraction $c_1 + \frac{1}{c_2 +} \frac{1}{c_3 +} \cdots$ form an *increasing* series of rational fractions continually approaching to the value of the complete continued fraction; and the *even*-numbered convergents form a *decreasing* series having the same property.

450. *The fraction* $\frac{u_1 u_2}{v_1 v_2}$ *is greater than or less than* x^2 *according as* $\frac{u_1}{v_1}$ *is greater than or less than* $\frac{u_2}{v_2}$.

For (§ 449), $M > 1$, $u_2 > u_1$, and $v_2 > v_1$.

$$\therefore M^2 u_2 v_2 - u_1 v_1 > 0.$$

Hence, $(M^2u_2v_2 - u_1v_1)(u_1v_2 - u_2v_1) > 0$ or < 0,

that is, $$M^2u_1u_2v_2^2 + u_1u_2v_1^2 > \text{ or } < M^2u_2^2v_1v_2 + u_1^2v_1v_2,$$

and $$u_1u_2(Mv_2 + v_1)^2 > \text{ or } < v_1v_2(Mu_2 + u_1)^2,$$

and, therefore, $$\frac{u_1u_2}{v_1v_2} > \text{ or } < \left(\frac{Mu_2 + u_1}{Mv_2 + v_1}\right)^2$$

according as $$\frac{u_1}{v_1} \text{ is } > \text{ or is } < \frac{u_2}{v_2}.$$

But $$x = \frac{Mu_2 + u_1}{Mv_2 + v_1}.$$

$$\therefore \frac{u_1u_2}{v_1v_2} > \text{ or } < x^2 \text{ according as } \frac{u_1}{v_1} > \text{ or } < \frac{u_2}{v_2}.$$

Corollary. $\dfrac{u_1u_2 - x^2v_1v_2}{u_1v_2 - u_2v_1}$ *is always positive.*

451. *Any convergent* $\dfrac{u_1}{v_1}$ *is nearer the complete value* x *than any other fraction with smaller denominator.*

Let $\dfrac{a}{b}$ be a fraction in which $b < v_1$.

If $\dfrac{a}{b}$ is one of the convergents, $x \sim \dfrac{a}{b} > \dfrac{u_1}{v_1} \sim x$. (§ 449)

If $\dfrac{a}{b}$ is not one of the convergents, and is nearer to x than is $\dfrac{u_1}{v_1}$, then, since x lies between $\dfrac{u_1}{v_1}$ and $\dfrac{u_2}{v_2}$ (§ 443), $\dfrac{a}{b}$ must be nearer to $\dfrac{u_2}{v_2}$ than is $\dfrac{u_1}{v_1}$.

That is, $$\frac{a}{b} \sim \frac{u_2}{v_2} < \frac{u_1}{v_1} \sim \frac{u_2}{v_2}, \text{ or } \frac{v_2a \sim u_2b}{v_2b} < \frac{1}{v_1v_2}.$$

Since $b < v_1$, this would require that $v_2a \sim u_2b < 1$. But $v_2a \sim u_2b$ cannot be less than 1, for a, b, u_2, v_2 are all integers.

Hence, $\dfrac{u_1}{v_1}$ is nearer to x than is $\dfrac{a}{b}$.

Examples. Express $\sqrt{3}$ in the form of a continued fraction.

Let $$\sqrt{3} = 1 + \frac{1}{x} \text{ (since 1 is the greatest integer in } \sqrt{3}).$$

Then, $$\frac{1}{x} = \sqrt{3} - 1.$$

$$\therefore\ x = \frac{1}{\sqrt{3} - 1} = \frac{\sqrt{3} + 1}{2}.$$

Let $$\frac{\sqrt{3} + 1}{2} = 1 + \frac{1}{y}\left(\text{since 1 is the greatest integer in } \frac{\sqrt{3} + 1}{2}\right).$$

Then, $$\frac{1}{y} = \frac{\sqrt{3} + 1}{2} - 1 = \frac{\sqrt{3} - 1}{2}.$$

$$\therefore\ y = \frac{2}{\sqrt{3} - 1} = \frac{\sqrt{3} + 1}{1}.$$

Let $$\frac{\sqrt{3} + 1}{1} = 2 + \frac{1}{z}\left(\text{since 2 is the greatest integer in } \frac{\sqrt{3} + 1}{1}\right).$$

Then, $$\frac{1}{z} = \frac{\sqrt{3} + 1}{1} - 2 = \sqrt{3} - 1.$$

$$\therefore\ z = \frac{1}{\sqrt{3} - 1} = \frac{\sqrt{3} + 1}{2}.$$

This is the same as x above; hence, the quotients 1, 2 will be continually repeated.

$$\therefore\ \sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \text{etc.}}},$$

of which $\dfrac{1}{1 + \frac{1}{2}}$ will be continually repeated, and the whole expression may be written

$$1 + \frac{\dot{1}}{1+}\,\frac{\dot{1}}{2}.$$

The convergents of the continued fraction $\dfrac{\dot{1}}{1+}\,\dfrac{\dot{1}}{2}$ are

$$1,\ \tfrac{2}{3},\ \tfrac{3}{4},\ \tfrac{8}{11},\ \tfrac{11}{15},\ \tfrac{30}{41},\ \cdots$$

$\therefore$ the convergents to $\sqrt{3}$ are $1,\ 2,\ \tfrac{5}{3},\ \tfrac{7}{4},\ \tfrac{19}{11},\ \tfrac{26}{15},\ \tfrac{71}{41},\ \cdots$

This example shows how any pure quadratic surd may be converted into a non-terminating simple continued fraction.

The following is another example with the work of conversion exhibited in full in a convenient arrangement.

$$\sqrt{7} = 2 + \frac{\sqrt{7}-2}{1} = 2 + \frac{3}{\sqrt{7}+2} = 2 + \frac{1}{x_1}.$$

$$\therefore x_1 = \frac{\sqrt{7}+2}{3} = 1 + \frac{\sqrt{7}-1}{3} = 1 + \frac{2}{\sqrt{7}+1} = 1 + \frac{1}{x_2}.$$

$$\therefore x_2 = \frac{\sqrt{7}+1}{2} = 1 + \frac{\sqrt{7}-1}{2} = 1 + \frac{3}{\sqrt{7}+1} = 1 + \frac{1}{x_3}.$$

$$\therefore x_3 = \frac{\sqrt{7}+1}{3} = 1 + \frac{\sqrt{7}-2}{3} = 1 + \frac{1}{\sqrt{7}+2} = 1 + \frac{1}{x_4}.$$

$$\therefore x_4 = \frac{\sqrt{7}+2}{1} = 4 + \frac{\sqrt{7}-2}{1} = 4 + \frac{3}{\sqrt{7}+2} = 4 + \frac{1}{x_1}.$$

$$\therefore \sqrt{7} = 2 + \frac{\dot{1}}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{\dot{1}}{4}.$$

Quotients $=$ 1, 1, 1, 4, 1, 1, 1, 4.

Convergents $= \frac{1}{0}, \frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{8}{3}, \frac{37}{14}, \frac{45}{17}, \frac{82}{31}, \frac{127}{48}, \frac{590}{223}.$

452. A non-terminating simple continued fraction in which the denominators recur, and recur always in the same order, is called **a simple periodic continued fraction.**

453. *Every quadratic surd may be converted into a simple periodic continued fraction.*

It is sufficient to consider the case of a pure quadratic surd, as a mixed surd can always be reduced to a pure surd.

Let N be any given integer not a square, and let q_1 be the integer next less than $\sqrt{N}$, hence $\sqrt{N} - q_1 < 1$. Then, arranging as in the last example, we have

$$\sqrt{N} = q_1 + \frac{\sqrt{N}-q_1}{1} = q_1 + \frac{r_1}{\sqrt{N}+k_1},$$

in which $k_1 = q_1$ and $r_1 = N - q_1^2$;

$$\frac{\sqrt{N}+k_1}{r_1} = q_2 + \frac{\sqrt{N}-k_2}{r_1} = q_2 + \frac{r_2}{\sqrt{N}+k_2},$$

if $k_2 = r_1 q_2 - k_1$ and $r_2 = \dfrac{N - k_2^2}{r_1}$;

$$\frac{\sqrt{N}+k_2}{r_2}=q_3+\frac{\sqrt{N}-k_3}{r_2}=q_3+\frac{r_3}{\sqrt{N}+k_3},$$
$$\text{if } k_3=r_2q_3-k_2 \text{ and } r_3=\frac{N-k_3^2}{r_2};$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{\sqrt{N}+k_{n-1}}{r_{n-1}}=q_n+\frac{\sqrt{N}-k_n}{r_{n-1}}=q_n+\frac{r_n}{\sqrt{N}+k_n}, \qquad [1]$$
$$\text{if } k_n=r_{n-1}q_n-k_{n-1} \text{ and } r_n=\frac{N-k_n^2}{r_{n-1}}.$$

Now, the numbers $r_1, r_2, r_3, \cdots$ *and* $k_2, k_3, \cdots$ *are positive integers.*

For, let $\frac{u_{n-1}}{v_{n-1}}, \frac{u_n}{v_n}, \frac{u_{n+1}}{v_{n+1}}$ be the consecutive convergents corresponding to the partial quotients q_{n-1}, q_n, q_{n+1}. The complete quotient next after q_n is $\frac{\sqrt{N}+k_n}{r_n}$, and using this instead of q_n to form the next convergent, we obtain the complete value $\sqrt{N}$ of the continued fraction, instead of the convergent value $\frac{u_{n+1}}{v_{n+1}}$. (§ 445)

$$\therefore \sqrt{N}=\frac{\left(\frac{\sqrt{N}+k_n}{r_n}\right)u_n+u_{n-1}}{\left(\frac{\sqrt{N}+k_n}{r_n}\right)v_n+v_{n-1}}=\frac{(\sqrt{N}+k_n)u_n+r_nu_{n-1}}{(\sqrt{N}+k_n)v_n+r_nv_{n-1}}.$$

Equate rational and irrational parts of this equation.

Then, $$k_nu_n+r_nu_{n-1}=v_nN,$$

and $$k_nv_n+r_nv_{n-1}=u_n.$$

$$\therefore k_n=\frac{u_{n-1}u_n-v_{n-1}v_nN}{u_{n-1}v_n-u_nv_{n-1}},$$

and $$r_n=\frac{v_n^2N-u_n^2}{u_{n-1}v_n-u_nv_{n-1}}. \qquad [2]$$

Hence, by § 450, Corollary, and § 443, Corollary, k_n and r_n are both positive, and since $u_{n-1}v_n-u_nv_{n-1}=\pm 1$, they

[illegible] integral: that is, [illegible] integers. Now, [illegible] positive integers. [illegible] Therefore, k_2 and [illegible]

[illegible] continued [illegible]

[illegible] $k_1, k_2, \ldots$ [illegible] are posi- [illegible] the greatest [illegible] t.

[illegible] and [illegible] can [illegible]

[illegible]

$$\sqrt{N} = q_1 + \frac{1}{q_2 +}\,\frac{1}{q_3 +} \cdots \frac{1}{+ q_t +}\,\frac{1}{q_{t+1} +}\,\frac{1}{q_{t+2} +} \cdots \frac{1}{+ q_m},$$

in which q_t is not equal to q_m.

Let $$y = q_{t+1} + \frac{1}{q_{t+2} +}\,\frac{1}{q_{t+3} +} \cdots \frac{1}{+ q_m},$$

and let $\frac{u_n}{v_n}$ denote the nth convergent to $\sqrt{N}$.

Then, $$\sqrt{N} = q_1 + \frac{1}{q_2 +}\,\frac{1}{q_3 +} \cdots \frac{1}{+ q_t +}\,\frac{1}{y}$$

$$= q_1 + \frac{1}{q_2 +}\,\frac{1}{q_3 +} \cdots \frac{1}{+ q_t +}\,\frac{1}{q_{t+1} +} \cdots \frac{1}{+ q_m +}\,\frac{1}{y}.$$

$$\therefore\ \sqrt{N} = \frac{yu_t + u_{t-1}}{yv_t + v_{t-1}} = \frac{yu_m + u_{m-1}}{yv_m + v_{m-1}}. \qquad [1]$$

Eliminate y from these equations,

$$(v_t v_{m-1} - v_{t-1} v_m) N - (v_t u_{m-1} - v_{t-1} u_m + u_t v_{m-1} - u_{t-1} v_m)\sqrt{N} + u_t u_{m-1} - u_{t-1} u_m = 0. \qquad [2]$$

$$\therefore\ v_t u_{m-1} - v_{t-1} u_m + u_t v_{m-1} - u_{t-1} v_m = 0.$$

$$\therefore\ u_{m-1} v_{t-1}\left(\frac{v_t}{v_{t-1}} - \frac{u_m}{u_{m-1}}\right) + u_{t-1} v_{m+1}\left(\frac{u_t}{u_{t-1}} - \frac{v_m}{v_{m-1}}\right) = 0.$$

$$\therefore\ u_{m-1} v_{t-1}\left(q_t + \frac{v_{t-2}}{v_{t-1}} - q_m - \frac{u_{m-2}}{u_{m-1}}\right) + u_{t-1} v_{m-1}\left(q_t + \frac{u_{t-2}}{u_{t-1}} - q_m - \frac{v_{m-2}}{v_{m-1}}\right) = 0. \qquad [3]$$

Now, since $t > 1$, $\frac{u_{m-2}}{u_{m-1}}$ is a positive proper fraction, and $\frac{v_{t-2}}{v_{t-1}}$ is zero if $t = 2$, and is a positive proper fraction if $t > 2$. Hence, $\frac{v_{t-2}}{v_{t-1}} - \frac{u_{m-2}}{u_{m-1}}$ is a proper fraction, say $\pm f$. So also $\frac{u_{t-2}}{u_{t-1}} - \frac{v_{m-2}}{v_{m-1}}$ is a proper fraction, say $\pm f'$. Hence, [3] may be written

$$u_{m-1} v_{t-1}(q_t - q_m \pm f) + u_{t-1} v_{m-1}(q_t - q_m \pm f') = 0. \qquad [4]$$

Now, f and f' both being proper fractions and $q_t \backsim q_m$ being an integer, for q_t was assumed unequal to q_m, the numbers $q_t - q_m \pm f$ and $q_t - q_m \pm f'$ are both positive or both negative, and [4] becomes the sum of two positive numbers or of two negative numbers is equal to zero; but this is impossible. Therefore, t cannot be greater than 1.

If $t = 1$, then $u_{t-1} = 1$, $v_{t-1} = 0$, $u_t = q_1$, $v_t = 1$, and equation [2] becomes

$$v_{m-1}N - (u_{m-1} + q_1 v_{m-1} - v_m)\sqrt{N} + q_1 u_{m-1} - u_m = 0.$$

$$\therefore v_{m-1}N + q_1 u_{m-1} - u_m = 0, \qquad [5]$$

and

$$u_{m-1} + q_1 v_{m-1} - v_m = 0. \qquad [6]$$

$$\therefore \frac{u_{m-1}}{v_{m-1}} + q_1 = \frac{v_m}{v_{m-1}}.$$

$$\therefore 2q_1 + \frac{1}{q_2 +}\,\frac{1}{q_3 +}\cdots\frac{1}{+ q_{m-1}} = q_m + \frac{1}{q_{m-1} +}\,\frac{1}{q_{m-2} +}\cdots\frac{1}{+ q_2}.$$

(See § 444, Corollary.)

$$\therefore q_m = 2q_1, \quad q_{m-1} = q_2, \quad q_{m-2} = q_3, \quad \cdots$$

$$\therefore \sqrt{N} = q_1 + \frac{\dot{1}}{q_2 +}\,\frac{1}{q_3 +}\cdots\frac{1}{+ q_3 +}\,\frac{1}{q_2 +}\,\frac{\dot{1}}{2q_1}. \qquad [7]$$

455. Eliminating q_1 from equations [5] and [6], we obtain

$$u_{m-1}^2 - Nv_{m-1}^2 = u_{m-1}v_m - u_m v_{m-1} = (-1)^{m-1}. \qquad [8]$$

Now, $m - 1$ is the number of terms in the cycle in [7]. Therefore, $u_{m-1}^2 - Nv_{m-1}^2 = +1$ or -1, according as the number of terms is even or is odd, in the cycle of the simple periodic continued fraction into which $\sqrt{N}$ is convertible.

Let $c_1 = u_{m-1}$ and $s_1 = v_{m-1}$, that is, let $\frac{c_1}{s_1}$ be the convergent immediately preceding the partial quotient $2q_1$ in [7]; then equation [8] becomes

$$c_1^2 - Ns_1^2 = +1 \text{ or } -1.$$

Consider the case $c_1^2 - Ns_1^2 = +1$.

Let $$(c_1 + s_1\sqrt{N})^n = c_n + s_n\sqrt{N}.$$

Then, $$(c_1 - s_1\sqrt{N})^n = c_n - s_n\sqrt{N};$$

$$\therefore c_n^2 - Ns_n^2 = (c_1^2 - Ns_1^2)^n = 1. \qquad [A]$$

Also $$(c_m + s_m\sqrt{N})(c_n + s_n\sqrt{N}) = c_{m+n} + s_{m+n}\sqrt{N}. \qquad [B]$$

Multiply the factors on the left side of [B] and equate rational and irrational parts; then

$$\left.\begin{aligned} c_{m+n} &= c_m c_n + N s_m s_n \\ s_{m+n} &= s_m c_n + s_n c_m \end{aligned}\right\}. \qquad [C]$$

456. These equations give a very easy and rapid method of obtaining a close approximation to $\sqrt{N}$.

From the example on page 368, we find for $\sqrt{7}$

$$\frac{c_1}{s_1} = \frac{8}{3}.$$

$$\therefore \frac{c_2}{s_2} = \frac{64 + 7 \times 3^2}{2 \times 8 \times 3} = \frac{127}{48},$$

and $$\frac{c_6}{s_6} = \frac{127^3 + 3 \times 7 \times 127 \times 48^2}{3 \times 127^2 \times 48 + 7 \times 48^3} = \frac{8193151}{3096720}.$$

By § 447, the error of approximation is

$$< \frac{1}{4 \times 3096720^2} < \frac{3}{10^{14}}.$$

457. Compare equations [A], [B], and [C] with the trigonometrical equations:

$$\cos^2\alpha - (-1)\sin^2\alpha = 1,$$

$$(\cos\alpha + \sqrt{-1}\sin\alpha)(\cos\beta + \sqrt{-1}\sin\beta)$$
$$= \cos(\alpha + \beta) + \sqrt{-1}\sin(\alpha + \beta),$$

and $$\cos(\alpha + \beta) = \cos\alpha\cos\beta + (-1)\sin\alpha\sin\beta,$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha.$$

458. If $$c_1^2 - Ns_1^2 = -1,$$
then $$c_2^2 - Ns_2^2 = +1;$$
and, in general, $$c_{2n}^2 - Ns_{2n}^2 = +1,$$
$$c_{2n+1}^2 - Ns_{2n+1}^2 = -1.$$

459. Equation [2], § 453, gives
$$u_n^2 - Nv_n^2 = (-1)^n r_{n+1},$$
and therefore, if
$$u_{n+mc} + v_{n+mc}\sqrt{N} = (u_n + v_n\sqrt{N})(c_m + s_m\sqrt{N}),$$
then $$u_{n+mc}^2 - Nv_{n+mc}^2 = (-1)^{m+n} r_{n+1}.$$

460. If g is the H.C.F. of mg and $a^2 - N$, to reduce the mixed surd $\dfrac{a+\sqrt{N}}{mg}$ to a simple periodic continued fraction, it is sufficient to reduce $\dfrac{am+\sqrt{m^2N}}{m^2g}$ to a continued fraction by the method of § 451.

Thus, $$\frac{3+\sqrt{7}}{8} = \frac{12+\sqrt{112}}{32} = \frac{1}{12-\sqrt{112}}$$
$$= \frac{1}{1+}\,\frac{1}{2+}\,\frac{\dot{1}}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{20+}\,\frac{1}{1+}\,\frac{\dot{1}}{1}.$$

461. The value of a simple periodic continued fraction can be expressed as the root of a quadratic equation.

Find the surd value of $\dfrac{\dot{1}}{1+}\,\dfrac{\dot{1}}{2}.$

Let x be the value of the continued fraction.

Then, $$x = \cfrac{1}{1+\cfrac{1}{2+x}} = \frac{2+x}{3+x}.$$
$$\therefore x^2 + 2x = 2.$$
$$\therefore x = -1 + \sqrt{3}.$$

We take the + sign since x is evidently positive.

It is not true, however, that a value can be determined for *any* periodic continued fraction; for if we assume

$$x = \frac{1}{1-}\,\frac{1}{1-}\,\frac{1}{1-}\,\frac{1}{1-}\cdots = \frac{1}{1-x},$$

we obtain
$$x^2 - x + 1 = 0.$$
$$\therefore x = \frac{1 \pm \sqrt{-3}}{2},$$

which is absurd. The continued fraction $\frac{1}{1-}\,\frac{1}{1-}\,\frac{1}{1-}\cdots$ is not convergent, as may be seen on attempting to form the principal convergents to it; these are

$$1,\quad \infty,\quad 0,\quad 1,\quad \infty,\quad 0.$$

462. Exponential Equations. An exponential equation can be solved by continued fractions.

Solve by continued fractions $10^x = 2$.

Let
$$x = 0 + \frac{1}{y}.$$

Then,
$$10^{\frac{1}{y}} = 2,$$
or
$$10 = 2^y.$$
$$\therefore y = 3 + \frac{1}{z} \text{ (since 10 lies between } 2^3 \text{ and } 2^4\text{).}$$

Then,
$$10 = 2^{3+\frac{1}{z}} = 2^3 \times 2^{\frac{1}{z}}.$$
$$\therefore 2^{\frac{1}{z}} = \tfrac{10}{8} = \tfrac{5}{4}.$$
$$\therefore 2 = (\tfrac{5}{4})^z.$$
$$\therefore z = 3 + \frac{1}{u} \text{ (since 2 lies between } (\tfrac{5}{4})^3 \text{ and } (\tfrac{5}{4})^4\text{).}$$

Then,
$$2 = (\tfrac{5}{4})^{3+\frac{1}{u}} = (\tfrac{5}{4})^3 \times (\tfrac{5}{4})^{\frac{1}{u}}.$$
$$\therefore (\tfrac{5}{4})^{\frac{1}{u}} = \tfrac{128}{125}.$$
$$\therefore \tfrac{5}{4} = (\tfrac{128}{125})^u.$$

The greatest integer in u is found to be 9.

Hence,
$$x = 0 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{9 + \text{etc.}}}}$$

The successive convergents are $\frac{1}{3}$, $\frac{3}{10}$, $\frac{28}{93}$, etc.
The last gives $x = \frac{28}{93} = 0.3010$, approximately.

NOTE. Observe that by the above process we have calculated the common logarithm of 2. By § 445, the error, when 0.3010 is taken for the common logarithm of 2, is considerably less than $\frac{1}{(93)^2}$, that is, considerably less than 0.00011; so that 0.3010 is certainly correct to three places of decimals, and probably correct to four places.

Logarithms are, however, much more easily calculated by the use of series, as shown in Chapter XXV.

Exercise 66

1. Find continued fractions for $\frac{133}{157}$, $\frac{159}{47}$, $\frac{103}{71}$, $\frac{67}{177}$, $\sqrt{5}$, $\sqrt{11}$, $4\sqrt{6}$; and find the fourth convergent to each.

2. Find continued fractions for $\frac{47}{357}$, $\frac{457}{904}$, $\frac{2066}{4626}$, $\frac{2991}{5658}$; and find the third convergent to each.

3. Find continued fractions for $\sqrt{21}$, $\sqrt{22}$, $\sqrt{33}$, $\sqrt{55}$.

4. Obtain convergents, with only two figures in the denominator, that approach nearest to the values of

$$\sqrt{7},\ \sqrt{10},\ \sqrt{15},\ \sqrt{17},\ \sqrt{18},\ \sqrt{20},\ 3 - \sqrt{5},\ 2 + \sqrt{11}.$$

5. If the pound troy is the weight of 22.8157 cubic inches of water, and the pound avoirdupois of 27.7274 cubic inches of water, find the fraction with denominator less than 100 which shall differ from their ratio by less than 0.0001.

6. The ratio of the diagonal to the side of a square being $\sqrt{2}$, find the fraction with denominator less than 100 which shall differ from their ratio by less than 0.0001.

7. Find the next convergent when the two preceding convergents are $\frac{3}{14}$ and $\frac{11}{51}$, and the next quotient is 5.

8. The ratio of the circumference of a circle to its diameter is approximately 3.14159265 : 1. Find the first three convergents to this ratio, and determine to how many decimal places each agrees with the true value.

9. In two scales of which the zero points coincide the distances between consecutive divisions of the one are to the corresponding distances of the other as 1 : 1.06577. Find what division points less than 100 most nearly coincide.

10. Find the surd values of

$$1+\frac{\dot{1}}{4+}\frac{\dot{1}}{2},\quad 3+\frac{\dot{1}}{1+}\frac{\dot{1}}{6},\quad \frac{\dot{1}}{3+}\frac{1}{1+}\frac{\dot{1}}{6},\quad 1+\frac{\dot{1}}{2+}\frac{1}{3+}\frac{\dot{1}}{4}.$$

11. Show that $\left(a+\frac{\dot{1}}{b+}\frac{\dot{1}}{a}\right)\left(\frac{\dot{1}}{b+}\frac{\dot{1}}{a}\right)=\frac{a}{b}.$

12. Show that the ratio of the diagonal of a cube to the edge may be nearly expressed by 97 : 56. Find the greatest possible value of the error made in taking this ratio for the true ratio.

13. Find a series of fractions converging to the ratio of 5 hours 48 minutes 51 seconds to 24 hours.

14. Find a series of fractions converging to the ratio of a cubic yard to a cubic meter, if a cubic yard is 0.76453 of a cubic meter.

CHAPTER XXVII

SCALES OF NOTATION

463. Definitions. Let any positive integer be selected as a **radix** or **base**; then any number may be expressed as an algebraic expression of which the terms are multiples of powers of the radix.

Any positive integer may be selected as the radix; and to each radix corresponds a **scale of notation**.

When we write numbers in any scale of notation, they are arranged by descending powers of the radix, and the powers of the radix are omitted, the *place* of each digit indicating of what power of the radix it is the coefficient.

Thus, in the scale of ten, 2356 stands for

$$2 \times 10^3 + 3 \times 10^2 + 5 \times 10 + 6;$$

in the scale of seven for

$$2 \times 7^3 + 3 \times 7^2 + 5 \times 7 + 6;$$

in the scale of r for

$$2r^3 + 3r^2 + 5r + 6.$$

464. Computation. Computations are made with numbers in any scale, by observing that one unit of any order is equal to the radix-number of units of the next lower order; and that the radix-number of units of any order is equal to one unit of the next higher order.

(1) Add 56,432 and 15,646 (scale of seven).

$$\begin{array}{r} 56432 \\ 15646 \\ \hline 105411 \end{array}$$

The process differs from that in the decimal scale only in that when a sum greater than *seven* is reached, we divide by *seven* (not ten), write the remainder, and carry the quotient to the next column.

(2) Subtract 34,561 from 61,235 (scale of eight).

$$\begin{array}{r} 61235 \\ \underline{34561} \\ 24454 \end{array}$$

When the number of any order of units in the minuend is less than the number of the corresponding order in the subtrahend, we increase the number in the minuend by eight instead of by ten, as in the common scale.

(3) Multiply 5732 by 428 (scale of nine).

$$\begin{array}{r} 5732 \\ \underline{428} \\ 51477 \\ 12564 \\ \underline{25238} \\ 2712127 \end{array}$$

We multiply the number of units in each order in the multiplicand by the number of units in each order in the multiplier, divide each time by *nine*, set down the remainder, and carry the quotient.

(4) Divide 2,712,127 by 5732 (scale of nine).

$$\begin{array}{r} 428 \\ 5732\,)\,2712127 \\ \underline{25238} \\ 17722 \\ \underline{12564} \\ 51477 \\ \underline{51477} \end{array}$$

The operations of multiplication and subtraction involved in this problem are precisely the same as in the decimal scale of notation, with the exception that the radix is 9 instead of 10.

465. Integers in Any Scale. *If* r *is any positive integer, any positive integer* N *may be expressed in the form*

$$\mathrm{N} = \mathrm{ar^n} + \mathrm{br^{n-1}} + \cdots + \mathrm{pr^2} + \mathrm{qr} + \mathrm{s},$$

in which the coefficients a, b, c, $\cdots$ *are positive integers, each less than* r.

For, divide N by r^n, the highest power of r contained in N, and let the quotient be a with the remainder N_1.

Then, $$N = ar^n + N_1.$$

In like manner,

$$N_1 = br^{n-1} + N_2; \quad N_2 = cr^{n-2} + N_3;$$

and so on.

By continuing this process a remainder s is at length reached which is less than r. So that,

$$N = ar^n + br^{n-1} + \cdots + pr^2 + qr + s.$$

Some of the coefficients s, q, p, $\cdots$ may vanish, and every coefficient is less than r; that is, the values of the coefficients may range from zero to $r-1$.

Hence, including zero, r digits are required to express numbers in the scale of r.

Express N in the form

$$ar^n + br^{n-1} + \cdots + pr^2 + qr + s,$$

and show how the digits a, b, $\cdots$ may be found.

If $\qquad N = ar^n + br^{n-1} + \cdots + pr^2 + qr + s,$

then $\qquad \dfrac{N}{r} = ar^{n-1} + br^{n-2} + \cdots + pr + q + \dfrac{s}{r}.$

That is, the remainder on dividing N by r is s, the last digit.

Let $\qquad N_1 = ar^{n-1} + br^{n-2} + \cdots + pr + q.$

Then, $\qquad \dfrac{N_1}{r} = ar^{n-2} + br^{n-3} + \cdots + p + \dfrac{q}{r}.$

That is, the remainder is q, the last but one of the digits.

466. Hence, **to express an integral number in the scale of r,**

Divide the number by the radix, then the quotient by the radix, and so on until a quotient less than the radix is reached. The successive remainders and the last quotient are the successive digits beginning with the units' place.

(1) Express 42,897 (scale of ten) in the scale of six.

$$\begin{array}{r}
6\,)\,\underline{42897} \\
6\,)\,\underline{7149}\cdots 3 \\
6\,)\,\underline{1191}\cdots 3 \\
6\,)\,\underline{198}\cdots 3 \\
6\,)\,\underline{33}\cdots 0 \\
5\cdots 3
\end{array}$$

Therefore, 42,897 (scale of ten) is expressed in the scale of six by 530,333.

(2) Change 37,214 from the scale of eight to the scale of nine.

$$\begin{array}{r}
9\,\underline{)\,37214} \\
9\,\underline{)\,3363}\cdots 1 \\
9\,\underline{)\,305}\cdots 6 \\
9\,\underline{)\,25}\cdots 8 \\
2\cdots 3
\end{array}$$

The radix is 8. Hence, the two digits on the left, 37, do not mean *thirty-seven*, but $3 \times 8 + 7$, or *thirty-one*, which contains 9 three times, with remainder 4.

The next partial dividend is $4 \times 8 + 2 = 34$, which contains 9 three times, with remainder 7; and so on.

Therefore, 37,214 (scale of eight) is expressed in the scale of nine by 23,861.

(3) In what scale is 140 (scale of ten) expressed by 352?

Let r be the radix; then, in the scale of ten,

$$140 = 3r^2 + 5r + 2, \text{ or } 3r^2 + 5r = 138.$$

Solving, we find $r = 6$.

The other value of r is negative and fractional, and therefore inadmissible, since the radix is always a positive integer.

467. Radix-Fractions. As in the decimal scale decimal fractions are used, so in any scale *radix-fractions* are used.

Thus, in the decimal scale, 0.2341 stands for

$$\frac{2}{10} + \frac{3}{10^2} + \frac{4}{10^3} + \frac{1}{10^4};$$

and in the scale of r it stands for

$$\frac{2}{r} + \frac{3}{r^2} + \frac{4}{r^3} + \frac{1}{r^4}.$$

(1) Express $\frac{245}{256}$ (scale of ten) by a radix-fraction in the scale of eight.

Assume $\quad \dfrac{245}{256} = \dfrac{a}{8} + \dfrac{b}{8^2} + \dfrac{c}{8^3} + \dfrac{d}{8^4} + \cdots$

Multiply by 8, $\quad 7\frac{21}{32} = a + \dfrac{b}{8} + \dfrac{c}{8^2} + \dfrac{d}{8^3} + \cdots$

Therefore, $\quad a = 7,$

and $\quad \dfrac{21}{32} = \dfrac{b}{8} + \dfrac{c}{8^2} + \dfrac{d}{8^3} + \cdots$

Multiply by 8, $\quad 5\frac{1}{4} = b + \dfrac{c}{8} + \dfrac{d}{8^2} + \cdots$

Therefore, $b = 5,$

and $\frac{1}{4} = \frac{c}{8} + \frac{d}{8^2} + \cdots$

Multiply by 8, $2 = c + \frac{d}{8} + \cdots$

Therefore, $c = 2,$

and $0 = d$, etc.

Therefore, $\frac{245}{256}$ (scale of ten) is expressed in the scale of eight by 0.752.

(2) Change 35.14 from the scale of eight to the scale of six.

We take the integral part and the fractional part separately.

Integral part:

$$\begin{array}{r|l} 6 & 35 \\ \hline & 4 \cdots 5. \end{array}$$

Fractional part: $\frac{1}{8} + \frac{4}{8^2} = \frac{12}{64} = \frac{3}{16}.$

This is reduced to a radix-fraction in the scale of six as in the margin.

$$\begin{array}{r}
3 \\
6 \\
16)\overline{18}(1 \\
\underline{16} \\
2 \\
6 \\
16)\overline{12}(0 \\
6 \\
16)\overline{72}(4 \\
\underline{64} \\
8 \\
6 \\
16)\overline{48}(3 \\
\underline{48} \\
\end{array}$$

Therefore, 35.14 (scale of eight) is expressed in the scale of six by 45.1043.

Exercise 67

1. Add 435, 624, 737 (scale of eight).
2. From 32,413 subtract 15,542 (scale of six).
3. Multiply 6431 by 35 (scale of seven).
4. Multiply 4685 by 3483 (scale of nine).
5. Divide 102,432 by 36 (scale of seven).
6. Find H.C.F. of 2541 and 3102 (scale of seven).
7. Extract the square root of 33,224 (scale of six).

8. Extract the square root of 300,114 (scale of five).

9. Change 624 from the scale of ten to the scale of five.

10. Change 3516 from the scale of seven to the scale of ten.

11. Change 3721 from the scale of eight to the scale of six.

12. Change 4535 from the scale of seven to the scale of nine.

13. Change 32.15 from the scale of six to the scale of nine.

14. Express $\frac{25}{128}$ (scale of ten) by a radix-fraction in the scale of four.

15. Express $\frac{43}{108}$ (scale of ten) by a radix-fraction in the scale of six.

16. Multiply 31.24 by 0.31 (scale of five).

17. In what scale is 21×36 equal to 746?

18. In what scale is the square of 23 expressed by 540?

19. In what scale are 212, 1101, 1220 in arithmetical progression?

20. Show that 1,234,321 is a perfect square in any scale (radix greater than four).

21. Which of the weights 1, 2, 4, 8, ⋯ pounds must be selected to weigh 345 pounds, only one weight of each kind being used?

22. Multiply 72,645 by 46,723 (scale of eight).

23. Divide 162,542 by 6522 (scale of seven).

24. A number of three digits in the scale of 7 is expressed in the scale of 9 by the same digits in reverse order. Find the number.

25. If two numbers are formed by the same digits in different orders, show that the difference between the numbers is divisible by $r - 1$.

CHAPTER XXVIII

THEORY OF NUMBERS

468. Definitions. In the present chapter, by the term *number* is meant *positive integer*. The terms *prime, composite,* are used in the ordinary arithmetical sense.

A *multiple* of a is a number that contains the factor a, and may be written ma.

An even number, since it contains the factor 2, may be written $2m$; an odd number may be written $2m + 1$, $2m - 1$, $2m + 3$, $2m - 3$, etc.

A number a is said *to divide* another number b when $\frac{b}{a}$ is an integer.

469. Resolution into Prime Factors. *A number can be resolved into prime factors in only one way.*

Let N be any number. Suppose $N = abc \cdots$, where $a, b, c, \cdots$ are prime numbers; suppose also $N = \alpha\beta\gamma \cdots$, where α, β, γ, $\cdots$ are prime numbers.

Then, $$abc \cdots = \alpha\beta\gamma \cdots$$

Hence, α must divide the product $abc \cdots$; but $a, b, c, \cdots$ are all prime numbers; hence, α must be equal to some one of them, a suppose.

Divide by a, $$bc \cdots = \beta\gamma \cdots; \text{ and so on.}$$

Hence, the factors in $\alpha\beta\gamma \cdots$ are equal to those in $abc \cdots$, and the theorem is proved.

470. Divisibility of a Product. I. *If a number* a *divides a product* bc, *and is prime to* b, *it must divide* c.

For, since a divides bc, every prime factor of a must be found in bc; but, since a is prime to b, no factor of a will be

found in b; hence, all the prime factors of a are found in c; that is, a divides c.

From this theorem it follows that:

II. *If a prime number* a *divides a product* bcde $\cdots$, *it must divide some factor of that product; and conversely.*

III. *If a prime number divides* b^n, *it must divide* b.

IV. *If* a *is prime to* b *and to* c, *it is prime to* bc.

V. *If* a *is prime to* b, *every power of* a *is prime to every power of* b.

471. *If* $\frac{a}{b}$, *a fraction in its lowest terms, is equal to another fraction* $\frac{c}{d}$, *then* c *and* d are *equimultiples of* a *and* b.

If $$\frac{a}{b}=\frac{c}{d}, \text{ then } \frac{ad}{b}=c.$$

Since b will not divide a, it must divide d; hence, d is a multiple of b.

Let $$d = mb, m \text{ being an integer.}$$

Since $\frac{a}{b}=\frac{c}{d}$, and $d=mb$, $\frac{a}{b}=\frac{c}{mb}$; therefore, $c = ma$.

Hence, c and d are equimultiples of a and b.

From the above theorem it follows that:

In the decimal scale of notation a common fraction in its lowest terms will produce a non-terminating decimal if its denominator contains any prime factor except 2 *and* 5.

For a terminating decimal is equivalent to a fraction with a denominator 10^n. Therefore, a fraction $\frac{a}{b}$ in its lowest terms cannot be equal to such a fraction, unless 10^n is a multiple of b. But 10^n, that is, $2^n \times 5^n$, contains no prime factors besides 2 and 5, and hence cannot be a multiple of b, if b contains any prime factors except 2 and 5.

472. Square Numbers. *If a square number is resolved into its prime factors, the exponent of each factor is even.*

For, if $$N = a^p \times b^q \times c^r \cdots$$
$$N^2 = a^{2p} \times b^{2q} \times c^{2r} \cdots$$

Conversely: A number that has the exponents of all its prime factors even is a perfect square; therefore,

To change any number to a perfect square,

Resolve the number into its prime factors, select the factors which have odd exponents, and multiply the given number by the product of these factors.

Thus, to find the least number by which 250 must be multiplied to make it a perfect square.

$250 = 2 \times 5^3$, in which 2 and 5 are the factors that have odd exponents.

Hence, the multiplier required is $2 \times 5 = 10$.

473. Divisibility of Numbers. I. *If two numbers* N *and* N′ *when divided by* a *have the same remainder, their difference is divisible by* a.

For, if N when divided by a has a quotient q and a remainder r, then
$$N = qa + r.$$

And, if N' when divided by a has a quotient q' and a remainder r, then
$$N' = q'a + r.$$

Therefore, $$N - N' = (q - q')a.$$

II. *If the difference between two numbers* N *and* N′ *is divisible by* a, *then* N *and* N′ *when divided by* a *have the same remainder.*

Let $$\frac{N - N'}{a} = m, \text{ where } m \text{ is an integer.}$$

Now, $$\frac{N}{a} = q + \frac{r}{a}, \text{ where } r < a,$$

and $$\frac{N'}{a} = q' + \frac{r'}{a}, \text{ where } r' < a.$$

Subtract, $$\frac{N - N'}{a} = q - q' + \frac{r - r'}{a}.$$

But $\frac{N - N'}{a}$ is an integer by hypothesis.

Therefore, $\frac{r - r'}{a}$ is an integer, or zero.

Now, $r - r' < r$ (r and r' being integers), and $r < a$.

$$\therefore r - r' < a.$$

Hence, a cannot divide $r - r'$. Therefore, $\frac{r - r'}{a}$ cannot be an integer, and hence must be zero.

Therefore, r must equal r'.

III. *If two numbers* N *and* N′ *when divided by a given number* a *have remainders* r *and* r′, *then* NN′ *and* rr′ *when divided by* a *have the same remainder.*

For, if $$N = qa + r,$$
and $$N' = q'a + r',$$
then $$NN' = qq'a^2 + qar' + q'ar + rr'$$
$$= (qq'a + qr' + q'r)a + rr'.$$

Therefore, NN' and rr' when divided by a have the same remainder.

Thus, 37 and 47 when divided by 7 have remainders 2 and 5.

Now, $37 \times 47 = 1739$, and $2 \times 5 = 10$.

The remainder when each of these two numbers is divided by 7 is 3.

From II it follows that, in the scale of ten,

1. *A number is divisible by* 2, 4, 8, $\cdots$ *if the numbers denoted by its last digit, last two digits, last three digits,* $\cdots$ *are divisible respectively by* 2, 4, 8, $\cdots$

2. *A number is divisible by* 5, 25, 125, $\cdots$ *if the numbers denoted by its last digit, last two digits, last three digits,* $\cdots$ *are divisible respectively by* 5, 25, 125, $\cdots$

3. *If from a number the sum of its digits is subtracted, the remainder is divisible by* 9.

For, if from a number expressed in the form

$$a + 10b + 10^2c + 10^3d + \cdots$$

$a + b + c + d + \cdots$ is subtracted, the remainder is

$$(10 - 1)b + (10^2 - 1)c + (10^3 - 1)d + \cdots$$

and $10 - 1$, $10^2 - 1$, $10^3 - 1$, $\cdots$ are each divisible by $10 - 1$, or 9.

Therefore, the remainder is divisible by 9.

4. *A number* N *may be expressed in the form* $9n + s$ (*if* s *denotes the sum of its digits*); *and* N *is divisible by* 3 *if* s *is divisible by* 3; *and by* 9 *if* s *is divisible by* 9.

5. *A number is divisible by* 11 *if the difference between the sum of its digits in the even places and the sum of its digits in the odd places is* 0 *or a multiple of* 11.

For, a number N expressed by digits (beginning from the right) a, b, c, d, $\cdots$ may be put in the form of

$$N = a + 10b + 10^2c + 10^3d + \cdots$$

$$\therefore N - a + b - c + d - \cdots = (10 + 1)b + (10^2 - 1)c + (10^3 + 1)d + \cdots$$

But $10 + 1$ is a factor of $10 + 1$, $10^2 - 1$, $10^3 + 1$, $\cdots$

Therefore, $N - a + b - c + d - \cdots$ is divisible by $10 + 1 = 11$.

Hence, the number N may be expressed in the form

$$11n + (a + c + \cdots) - (b + d + \cdots),$$

and is a multiple of 11, if $(a + c + \cdots) - (b + d + \cdots)$ is 0 or a multiple of 11.

474. Theorem. *The product of* r *consecutive integers is divisible by* $\underline{|r}$.

Represent by $P_{n,k}$ the product of k consecutive integers beginning with n.

Then, $\quad P_{n,k} = n(n + 1)\cdots(n + k - 1);$

$$P_{n+1,k+1} = (n + 1)(n + 2)\cdots(n + k)(n + k + 1)$$
$$= n(n + 1)(n + 2)\cdots(n + k)$$
$$+ (k + 1)(n + 1)(n + 2)\cdots(n + k).$$

$$\therefore P_{n+1,k+1} = P_{n,k+1} + (k + 1)P_{n+1,k}.$$

Assume, for the moment, that the product of any k consecutive integers is divisible by $\underline{|k}$.

Then, $\quad P_{n+1,k+1} = P_{n,k+1} + (k+1)M\underline{|k}$;

or, $\quad P_{n+1,k+1} = P_{n,k+1} + M\underline{|k+1}$,

where M is an integer.

Hence, if $P_{n,k+1}$ is divisible by $\underline{|k+1}$, $P_{n+1,k+1}$ is also divisible by $\underline{|k+1}$. But $P_{1,k+1}$ is divisible by $\underline{|k+1}$ since $P_{1,k+1} = \underline{|k+1}$. Therefore, $P_{2,k+1}$ is divisible by $\underline{|k+1}$; hence, $P_{3,k+1}$ is divisible by $\underline{|k+1}$; and so on.

Hence, the product of any $k+1$ consecutive integers is divisible by $\underline{|k+1}$, if the product of any k consecutive integers is divisible by $\underline{|k}$. The product of any 2 consecutive integers is divisible by $\underline{|2}$; hence, the product of any 3 consecutive integers is divisible by $\underline{|3}$; hence, the product of any 4 consecutive integers is divisible by $\underline{|4}$; and so on. Therefore, the product of any r consecutive integers is divisible by $\underline{|r}$.

475. Examples. (1) Show that every square number is of one of the forms $5n$, $5n-1$, $5n+1$.

Every number is of one of the forms:

$$5n-2,\ 5n-1,\ 5n,\ 5n+1,\ 5n+2.$$

Now, $\quad (5n \pm 2)^2 = 25n^2 \pm 20n + 4 = 5(5n^2 \pm 4n + 1) - 1$;

$\quad (5n \pm 1)^2 = 25n^2 \pm 10n + 1 = 5(5n^2 \pm 2n) + 1$;

and $\quad (5n)^2 = 25n^2 \quad = 5(5n^2)$.

Therefore, every square number is of one of the three forms:

$$5n,\ 5n-1,\ 5n+1.$$

Hence, in the scale of ten, every square number must end in 0, 1, 4, 5, 6, or 9.

(2) Show that $n^5 - n$ is divisible by 30 if n is even.

$$\begin{aligned} n^5 - n &= n(n-1)(n+1)(n^2+1) \\ &= n(n-1)(n+1)(n^2-4+5) \\ &= n(n-1)(n+1)[(n-2)(n+2)+5]. \end{aligned}$$

Now, $n(n-1)(n+1)$ is divisible by $\underline{|3}$. (§ 474)

One of the five consecutive numbers $n-2$, $n-1$, n, $n+1$, $n+2$ is divisible by 5.

If n, $(n-1)$, or $(n+1)$ is divisible by 5, the number on the right is divisible by 5. If $(n-2)$ or $(n+2)$ is divisible by 5, then $[(n-2)(n+2)+5]$ is divisible by 5, since 5 is divisible by 5.

Therefore, the number on the right is always divisible by 5.

Hence, $n^5 - n$ is divisible by $5 \times \underline{|3}$, that is, by 30.

Exercise 68

Find the least number by which each of the following must be multiplied that the product may be a square number:

1. 2625. 2. 3675. 3. 4374. 4. 74,088.

5. If m and n are positive integers, both odd or both even, show that $m^2 - n^2$ is divisible by 4.

6. Show that $n^2 - n$ is always even.

7. Show that $n^3 - n$ is divisible by 6 if n is even, and by 24 if n is odd.

8. Show that $n^5 - n$ is divisible by 240 if n is odd.

9. Show that $n^7 - n$ is divisible by 42 if n is even, and by 168 if n is odd.

10. Show that $n(n+1)(n+5)$ is divisible by 6.

11. Show that every square number is of one of the forms $3n$, $3n+1$.

12. Show that every cube number is of one of the forms $9n$, $9n-1$, $9n+1$.

13. Show that every cube number is of one of the forms $7n$, $7n-1$, $7n+1$.

14. Show that every number which is both a square and a cube is of the form $7n$ or $7n+1$.

15. Show that in the scale of ten every perfect fourth power ends in one of the figures 0, 1, 5, 6.

CHAPTER XXIX

DETERMINANTS

476. Origin. If we solve the two simultaneous equations

$$a_1x + b_1y = c_1,$$
$$a_2x + b_2y = c_2,$$

we obtain

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Similarly, from the three simultaneous equations

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3,$$

we obtain

$$x = \frac{d_1b_2c_3 - d_1b_3c_2 + d_2b_3c_1 - d_2b_1c_3 + d_3b_1c_2 - d_3b_2c_1}{a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1},$$

with similar expressions for y and z.

The numerators and denominators of these fractions are examples of expressions which often occur in algebraic work, and for which it is therefore convenient to have a special name. Such expressions are called **determinants.**

477. Definitions. Determinants are usually written in a compact form, called the *square form.*

Thus, $a_1b_2 - a_2b_1$ is written $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$,

and $a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$ is written

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

This square form is sometimes written in a still more abbreviated form. Thus, the last two determinants are written $|a_1\ b_2|$ and $|a_1\ b_2\ c_3|$. This last notation should, however, always suggest the square form. In any problem it generally is advisable to write this abbreviated form in the complete square form.

The individual symbols a_1, a_2, b_1, b_2, $\cdots$ are called **elements.**

A horizontal line of elements is called a **row**; a vertical line a **column.**

The two lines a_1, b_2, c_3 and a_3, b_2, c_1 are called **diagonals**; the first the **principal diagonal,** the second the **secondary diagonal.**

The **order** of a determinant is the number of elements in a row or column.

Thus, the last two determinants are of the second and third orders respectively.

The expression of which the square form is an abbreviation is called the **expansion** of the determinant.

The several terms of the expansion are called **terms** of the determinant.

Thus, the expansion of $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ is $a_1b_2 - a_2b_1$.

REMARK. By some writers *constituent* is used where we use *element,* and *element* where we use *term.*

478. General Definition. In general, a determinant of the nth order is an expression involving n^2 elements arranged in n rows of n elements each.

479. Inversions of Order. In any arrangement of the letters of a determinant the occurrence of any one of them before another which precedes it in the principal diagonal is called an *inversion of order.*

Thus, if 1, 2, 3, 4, 5 is the order in the principal diagonal, in the order 2, 3, 5, 1, 4 there are four inversions : 2 before 1, 3 before 1, 5 before 1, 5 before 4.

Similarly, if a, b, c, d is the order in the principal diagonal, in the order b, d, a, c there are three inversions: b before a, d before a, d before c.

480. In any arrangement of integers (or letters) let two adjacent integers (or letters) be interchanged; then the number of inversions is either increased or diminished by one.

For example, in the arrangement 6 2 [5 1] 4 3 7 interchange 5 and 1.

We now have 6 2 [1 5] 4 3 7.

The inversions of 5 and 1 with the integers before the group are the same in each arrangement.

The inversions of 5 and 1 with the integers after the group are the same in each series.

In the first arrangement 5 1 is an inversion; in the second series 1 5 is not an inversion.

Hence, the interchanging of 5 and 1 diminishes the number of inversions by one.

Similarly for any other case.

481. Signs of the Terms. *The principal diagonal term always has a + sign.*

To find the sign of any other term:

Add the number of inversions among the letters, and the number of inversions among the subscripts. If the total number is even, *the sign of the term is* +; *if* odd, −.

Thus, in the determinant $|a_1\ b_2\ c_3\ d_4|$ consider the term $c_2a_3d_4b_1$. There are in $c\ a\ d\ b$ three inversions; in 2 3 4 1 three inversions; the total is six, an even number, and the sign of the term is +.

482. Special Rules. In practice the sign of a term is easily found by one of the following special rules:

I. *Write the elements of the term in the natural order of letters; if the number of inversions among the subscripts is even, the sign of the term is* +; *if odd*, −.

II. *Write the elements in the natural order of subscripts; if the number of inversions among the letters is even, the sign of the term is* +; *if odd*, −.

Thus, in the determinant $|a_1\ b_2\ c_3\ d_4|$ consider the term $c_2a_3d_4b_1$. Writing the elements in the order of letters, we have $a_3b_1c_2d_4$. There are two inversions, *viz.*, 3 before 1, and 3 before 2, and the sign of the term is +. Or, write the elements in the order of subscripts, $b_1c_2a_3d_4$.

There are two inversions, *viz.*, b before a, and c before a, and the sign of the term is $+$.

That these special rules give the same sign as the general rule of § 481 may be seen as follows:

Consider the term $c_2a_3d_4b_1$. Its sign is determined by the total number of inversions in the two series $\begin{matrix} c\ a\ d\ b \\ 2\ 3\ 4\ 1 \end{matrix}$. Bring a_3 to the first position; this interchanges in the two series c and a, 2 and 3. In each series the number of inversions is increased or diminished by one (§ 480), and the total is, therefore, increased or diminished by an even number.

Interchange b_1 and d_4, then interchange b_1 and c_2; this brings b_1 to the second place, and the letters into the natural order. As before, the total number of inversions is changed by an even number.

The term is now written $a_3b_1c_2d_4$, and the number of inversions differs by an even number from that found by the general rule of § 481. Hence, the sign given by I agrees with the sign given by the general rule.

483. The Expansion. The expansion of any determinant may be found by forming all the possible products by taking one element, *and only one*, from each row, and one element, *and only one*, from each column, and prefixing to each product the proper sign.

The number of terms in the expansion is

$$1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n = \lfloor n.$$

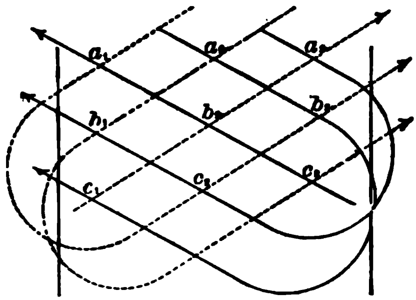
For, to form the expansion of a determinant of the nth order we make all the possible arrangements of n elements, taking all of them in each arrangement (§ 339).

484. If all the elements in any row or column are zero, the determinant is zero. For every term contains one of the zeros from this row or column (§ 483), and therefore every term of the determinant is zero.

A determinant is unchanged if the rows are changed to columns and the columns to rows. For the rules (§§ 478–483) are unchanged if row is changed to column and column to row.

Thus, $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

485. A determinant of the *third order* may be conveniently expanded as follows:



Three elements connected by a full line form a positive term; three elements connected by a dotted line form a negative term. The expansion obtained from the diagram is

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1,$$

which agrees with § 477.

There is no simple rule for expanding determinants of orders higher than the third.

Exercise 69

Prove the following relations by expanding:

1. $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \equiv \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv - \begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix} \equiv \begin{vmatrix} b_2 & b_1 \\ a_2 & a_1 \end{vmatrix}.$

2. $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_3 & a_2 & a_1 \\ c_3 & c_2 & c_1 \\ b_3 & b_2 & b_1 \end{vmatrix} \equiv - \begin{vmatrix} b_1 & c_1 & a_1 \\ b_3 & c_3 & a_3 \\ b_2 & c_2 & a_2 \end{vmatrix}.$

Find the value of:

3. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & 4 & 5 \end{vmatrix}.$ 4. $\begin{vmatrix} 3 & 2 & 4 \\ 7 & 6 & 1 \\ 5 & 3 & 8 \end{vmatrix}.$ 5. $\begin{vmatrix} 4 & 5 & 2 \\ -1 & 2 & -3 \\ 6 & -4 & 5 \end{vmatrix}.$

Find the value of:

6. $\begin{vmatrix} 5 & 8 & 7 \\ 6 & 4 & 4 \\ 5 & 7 & 9 \end{vmatrix}$. 7. $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{vmatrix}$. 8. $\begin{vmatrix} 9 & 6 & 3 \\ 5 & 7 & 7 \\ 8 & 4 & 2 \end{vmatrix}$.

9. Count the inversions in the arrangement:

5 4 1 3 2. 7 5 1 4 3 6 2. $d\ a\ c\ e\ b$.

4 1 5 2 3. 6 5 4 2 1 3 7. $c\ e\ b\ d\ a$.

10. In the determinant $|a_1\ b_2\ c_3\ d_4\ e_5|$ find the signs of the following terms:

$a_1b_4c_5d_3e_2$. $a_3b_1c_5d_4e_2$. $e_1c_4a_2b_5d_3$.

$a_2b_5c_3d_1e_4$. $b_4c_5a_1e_3d_2$. $c_1a_3b_5e_4d_2$.

11. Write, with their proper signs, all the terms of the determinant $|a_1\ b_2\ c_3\ d_4|$.

12. Write, with their proper signs, all the terms of the determinant $|a_1\ b_2\ c_3\ d_4\ e_5|$ which contain both a_1 and b_4; all the terms which contain both b_3 and e_2.

Expand:

13. $\begin{vmatrix} a & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & a & a & b \\ 0 & b & b & a \end{vmatrix}$. 14. $\begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ a & a & b & b \\ b & b & a & a \end{vmatrix}$. 15. $\begin{vmatrix} a & b & c & 0 \\ c & a & b & 0 \\ b & c & a & 0 \\ a & b & c & 1 \end{vmatrix}$.

496. Number of Terms. Consider a determinant of the nth order.

In forming a term we can take from the first row any one of n elements; from the second row any one of $n-1$ elements; and so on. From the last row we can take only the one remaining element.

Hence, the full number of terms is $n(n-1)\cdots 1$, or $\lfloor n$.

487. Interchange of Columns or Rows. *If two adjacent columns or two adjacent rows of a determinant Δ are interchanged, the determinant thus obtained is $-\Delta$.*

For example, consider the determinants

$$\Delta \equiv \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}, \qquad \Delta' \equiv \begin{vmatrix} a_1 & a_3 & a_2 & a_4 \\ b_1 & b_3 & b_2 & b_4 \\ c_1 & c_3 & c_2 & c_4 \\ d_1 & d_3 & d_2 & d_4 \end{vmatrix}.$$

The individual elements in any row or column of Δ' are the same as those of some row or column of Δ, the only difference being in the *arrangement* of the elements. Since every term of each determinant contains one, and only one, element from each row and column, every term of Δ' must, disregarding the sign, be a term of Δ.

Now the sign of any particular term of Δ' is found from an arrangement (§ 482, I) in which 3 2 is the natural order. The sign of the term of Δ which contains the same elements is found from an arrangement in which 3 2 is regarded as an inversion. Consequently, every term which in Δ' has a $+$ sign has in Δ a $-$ sign, and *vice versa* (§ 480).

Therefore, $$\Delta' \equiv -\Delta.$$

Similarly if any two adjacent columns or two adjacent rows of *any* determinant are interchanged.

488. *In any determinant Δ, if a particular column is carried over* m *columns, the determinant obtained is* $(-1)^m\Delta$.

For, successively interchange the column in question with the adjacent column until it occupies the desired position. There are m interchanges made; hence, there are m changes of sign (§ 487). We may make each change of sign by multiplying Δ by -1. Therefore, the new determinant is $(-1)^m\Delta$.

Similarly for a particular row.

489. *In any determinant Δ, if any two columns are interchanged, the determinant thus obtained is $-\Delta$.*

Let there be m columns between the columns in question.

Bring the second column before the first. The second column is carried over $m+1$ columns, and the determinant obtained is $(-1)^{m+1}\Delta$ (§ 488).

Bring the first column to the original position of the second. The first column is carried over m columns, and the determinant obtained is $(-1)^m(-1)^{m+1}\Delta$, or $(-1)^{2m+1}\Delta$.

Since $2m+1$ is always an odd number, $(-1)^{2m+1}\Delta = -\Delta$.

Similarly for two rows.

Thus, $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv - \begin{vmatrix} a_3 & a_2 & a_1 \\ b_3 & b_2 & b_1 \\ c_3 & c_2 & c_1 \end{vmatrix} \equiv \begin{vmatrix} a_3 & a_2 & a_1 \\ c_3 & c_2 & c_1 \\ b_3 & b_2 & b_1 \end{vmatrix}.$$

490. Useful Properties. *If two columns of a determinant are identical, the determinant vanishes.*

For, let Δ represent the determinant.

Interchanging the two identical columns ought to change Δ into $-\Delta$ (§ 489). But since the two columns are identical, the determinant is unchanged.

$$\therefore \Delta \equiv -\Delta, \quad 2\Delta \equiv 0, \quad \Delta \equiv 0.$$

Similarly if two rows are identical.

491. *If all the elements in any column are multiplied by any number* m, *the determinant is multiplied by* m.

For, every term contains one, and only one, element from the column in question. Hence every term, and consequently the whole determinant, is multiplied by m.

Similarly for a row.

Thus, $$\begin{vmatrix} ma_1 & a_2 & a_3 \\ mb_1 & b_2 & b_3 \\ mc_1 & c_2 & c_3 \end{vmatrix} \equiv m \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv \begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Again, $\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} \equiv \frac{1}{abc}\begin{vmatrix} abc & a^2 & a^3 \\ bca & b^2 & b^3 \\ cab & c^2 & c^3 \end{vmatrix} \equiv \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}.$

492. If each of the elements in a column or row is the sum of two numbers, the determinant may be expressed as the sum of two determinants.

Thus, $\begin{vmatrix} a_1+\alpha & a_2 & a_3 \\ b_1+\beta & b_2 & b_3 \\ c_1+\gamma & c_2 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha & a_2 & a_3 \\ \beta & b_2 & b_3 \\ \gamma & c_2 & c_3 \end{vmatrix}.$

For, consider any term, as $(a_1+\alpha)b_2c_3$. This may be written $a_1b_2c_3+\alpha b_2c_3$. Hence, every term of the first determinant is the sum of a term of the second determinant and a term of the third determinant. Consequently, the first determinant is the sum of the other two determinants.

Similarly for any other case.

493. If the elements in any column or row are multiplied by any number m, and added to, or subtracted from, the corresponding elements in any other column or row, the determinant is unchanged.

Thus, $\begin{vmatrix} a_1 \pm ma_2 & a_2 & a_3 \\ b_1 \pm mb_2 & b_2 & b_3 \\ c_1 \pm mc_2 & c_2 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \pm \begin{vmatrix} ma_2 & a_2 & a_3 \\ mb_2 & b_2 & b_3 \\ mc_2 & c_2 & c_3 \end{vmatrix}.$

The last determinant may be written

$\pm m\begin{vmatrix} a_2 & a_2 & a_3 \\ b_2 & b_2 & b_3 \\ c_2 & c_2 & c_3 \end{vmatrix}$, and therefore vanishes (§ 490).

Hence, we have only the first determinant on the right-hand side.

Similarly for any other case.

This process may be applied simultaneously to two or more columns or rows; but in this case care must be taken not to make two columns or rows identical (§ 490).

This last property is of great use in reducing determinants to simpler forms.

494. Examples. (1) $\begin{vmatrix} b+c & a & 1 \\ c+a & b & 1 \\ a+b & c & 1 \end{vmatrix} \equiv \begin{vmatrix} b+c+a & a & 1 \\ c+a+b & b & 1 \\ a+b+c & c & 1 \end{vmatrix}$

$$\equiv (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} \equiv 0.$$

Begin by adding the second column to the first.

(2) $\begin{vmatrix} 14 & 15 & 11 \\ 21 & 22 & 16 \\ 23 & 29 & 17 \end{vmatrix} = \begin{vmatrix} 3 & 4 & 11 \\ 5 & 6 & 16 \\ 6 & 12 & 17 \end{vmatrix} = 2\begin{vmatrix} 3 & 2 & 11 \\ 5 & 3 & 16 \\ 6 & 6 & 17 \end{vmatrix}$

$$= 2\begin{vmatrix} 3 & 2 & 2 \\ 5 & 3 & 1 \\ 6 & 6 & -1 \end{vmatrix} = 2(19) = 38.$$

Begin by subtracting the third column from the first and second columns. Then take out the factor 2 (§ 491), subtract 3 times the first column from the third, and expand the result by § 485.

495. Factoring of Determinants. *If a determinant vanishes when for any element* a *we put another element* b, *then* a − b *is a factor of the determinant.*

For, the expansion contains only positive integral powers of the several elements, and we can write

$$\Delta \equiv A_0 + A_1 a + A_2 a^2 + A_3 a^3 + \cdots, \qquad [1]$$

where $A_0, A_1, A_2, A_3, \cdots$ are expressions that do not involve a, and consequently remain unchanged when we put b for a.

Put b for a. Since Δ becomes 0 by hypothesis,

$$0 \equiv A_0 + A_1 b + A_2 b^2 + A_3 b^3 + \cdots \qquad [2]$$

Subtract [2] from [1],

$$\Delta \equiv A_1(a-b) + A_2(a^2-b^2) + A_3(a^3-b^3) + \cdots$$

Since $a - b$ is a factor of each term of the expansion, $a - b$ is a factor of the determinant.

The theorem also holds true when a and b are not elements, provided a and b enter into the expansion in positive integral powers only.

By the principle just proved, and the principle of § 490, we can resolve into factors many determinants without expanding them.

(1) Resolve into factors $\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$.

The determinant vanishes when $a = b$, when $a = c$, and when $b = c$. Hence, $a - b$, $b - c$, and $c - a$ are factors. Δ is of the third degree in a, b, c, and these are easily seen to be all the factors. It remains to determine the sign before the product.

In Δ, as given, a^2b is $+$; in the product $(a - b)(b - c)(c - a)$ the term a^2b is $-$. Hence,

$$\Delta \equiv -(a - b)(b - c)(c - a).$$

(2) Resolve into factors $\begin{vmatrix} a^2 & a & b + c \\ b^2 & b & c + a \\ c^2 & c & a + b \end{vmatrix}$.

As in the last example $a - b$, $b - c$, $c - a$ are found to be factors. There is one other factor of the first degree.

To the third column add the second; the result may be written

$$(a + b + c)\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix};$$

or, by Example (1), $-(a + b + c)(a - b)(b - c)(c - a)$.

Exercise 70

Show that:

1. $\begin{vmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{vmatrix} \equiv 2\,abc.$

2. $\begin{vmatrix} b + c & a & a \\ b & c + a & b \\ c & c & a + b \end{vmatrix} \equiv 4\,abc.$

3. $\begin{vmatrix} 1 & a^2 & a^3 & a^4 \\ 1 & b^2 & b^3 & b^4 \\ 1 & c^2 & c^3 & c^4 \\ 1 & d^2 & d^3 & d^4 \end{vmatrix} \equiv \begin{vmatrix} bcd & a & a^2 & a^3 \\ cda & b & b^2 & b^3 \\ dab & c & c^2 & c^3 \\ abc & d & d^2 & d^3 \end{vmatrix}.$

4. $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix} \equiv \begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix}.$

Find the value of:

5. $\begin{vmatrix} 20 & 15 & 25 \\ 17 & 12 & 22 \\ 19 & 20 & 16 \end{vmatrix}.$ 6. $\begin{vmatrix} 3 & 23 & 13 \\ 7 & 53 & 30 \\ 9 & 70 & 39 \end{vmatrix}.$ 7. $\begin{vmatrix} 22 & 29 & 27 \\ 25 & 23 & 30 \\ 28 & 26 & 24 \end{vmatrix}.$

Resolve into simplest factors:

8. $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}.$ 9. $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}.$ 10. $\begin{vmatrix} a^3 & bc & 1 \\ b^3 & ca & 1 \\ c^3 & ab & 1 \end{vmatrix}.$

11. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$ 12. $\begin{vmatrix} 1 & a & a^2 & a^4 \\ 1 & b & b^2 & b^4 \\ 1 & c & c^2 & c^4 \\ 1 & d & d^2 & d^4 \end{vmatrix}.$ 13. $\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}.$

14. If all the elements on one side of a diagonal term are zeros, show that the expansion reduces to this term.

Show that:

15. $\begin{vmatrix} a^2 - bc & a & 1 \\ b^2 - ca & b & 1 \\ c^2 - ab & c & 1 \end{vmatrix} \equiv 0.$

16. $\begin{vmatrix} a + 2b & a + 4b & a + 6b \\ a + 3b & a + 5b & a + 7b \\ a + 4b & a + 6b & a + 8b \end{vmatrix} \equiv 0.$

17. $$\begin{vmatrix} b^2+c^2 & ba & ca \\ ab & c^2+a^2 & cb \\ ac & bc & a^2+b^2 \end{vmatrix} \equiv 4\,a^2b^2c^2.$$

18. $$\begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix} \equiv 2\,abc\,(a+b+c)^3.$$

19. $$\begin{vmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{vmatrix} \equiv x^4 + 10\,x^3.$$

20. $$\begin{vmatrix} a^2+1 & ba & ca & da \\ ab & b^2+1 & cb & db \\ ac & bc & c^2+1 & dc \\ ad & bd & cd & d^2+1 \end{vmatrix} \equiv a^2+b^2+c^2+d^2+1.$$

496. Minors. If one row and one column of a determinant are erased, a new determinant of order one lower than the given determinant is obtained. This determinant is called a **first minor** of the given determinant.

Similarly, by erasing two rows and two columns, we obtain a **second minor**; and so on.

Thus, in the determinant $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, by erasing the second row and the third column, we obtain the first minor $\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$. This minor is said to correspond to the element b_3, and is generally represented by Δ_{b_3}; so that, in this case, $\Delta_{b_3} \equiv \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$.

In general, to every element corresponds a first minor obtained by erasing the row and column in which the given element stands.

497. ***If all the elements of the first row after the first element are zeros, the determinant reduces to $a_1\Delta_{a_1}$.***

Consider the determinant $\Delta \equiv \begin{vmatrix} a_1 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$.

Every term of Δ contains one, and only one, element from the first row; and each term that does not contain a_1 contains one of the zeros, and therefore vanishes. Each term that contains a_1 contains no other element from the first row or column, and, consequently, contains one, and only one, element from each row and column of the determinant

$$\begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix}, \text{ or } \Delta_{a_1}.$$

Hence, disregarding the sign, each term of Δ consists of a_1 multiplied into a term of Δ_{a_1}.

Take any particular term of Δ, as $a_1b_4c_3d_2$; the sign is fixed (§ 482, I) by the number of inversions in the series 1 4 3 2; the sign of the term $b_4c_3d_2$ of Δ_{a_1} is fixed by the number of inversions in the series 4 3 2. Adding a_1 makes no new inversions among either the letters or the subscripts. Consequently, the sign of the term in Δ is the same as the sign of the term in $a_1\Delta_{a_1}$.

Since this is true of every term of Δ, we have

$$\Delta \equiv a_1\Delta_{a_1}.$$

Similarly for any determinant of like form.

498. Terms containing an Element. By § 486 the sum of the terms that contain a_1 may be written $a_1\Delta_{a_1}$. For, no one of the terms that contain a_1 can contain any one of the elements $a_2, a_3, a_4, \cdots$, and these terms are therefore unchanged if for $a_2, a_3, a_4, \cdots$ in the given determinant we put zeros.

If we carry the second column over the first, the determinant is changed to $-\Delta$. By § 496 the sum of the terms of $-\Delta$ that contain a_2 is $a_2\Delta_{a_2}$, and the sum of the corresponding terms of Δ is, therefore, $-a_2\Delta_{a_2}$.

In general, for the element of the pth row and qth column we carry the pth row over $p-1$ rows, and the qth column over $q-1$ columns, in order to bring the element in question to the first row and first column. The new determinant is Δ if $p+q-2$ is *even*, and is $-\Delta$ if $p+q-2$ is *odd* (§ 488). Consequently, the sum of the terms of Δ that contain the element of the pth row and qth column is the product of that element by its minor, the sign being $+$ if $p+q$ is *even*, and $-$ if $p+q$ is *odd*.

Thus, in $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$ we find that the sum of the terms which contain c_3 is $c_3\Delta_{c_3}$.

Here, $p=3$, $q=3$, and $p+q$ is *even*.

499. Co-Factors. Since every term contains one element from each row and column, if we add the sum of terms containing a_1, the sum of the terms containing a_2, and so on, we shall obtain the whole expansion of the given determinant.

Thus, in the determinant $|a_1\, b_2\, c_3\, d_4|$,

$$\Delta \equiv a_1\Delta_{a_1} - a_2\Delta_{a_2} + a_3\Delta_{a_3} - a_4\Delta_{a_4}.$$

The expressions Δ_{a_1}, $-\Delta_{a_2}$, Δ_{a_3}, $-\Delta_{a_4}$ are called the **co-factors** of the several elements a_1, a_2, a_3, a_4 and are generally represented by A_1, A_2, A_3, A_4.

Hence, in the case of $|a_1\, b_2\, c_3\, d_4|$, we may write

$$\begin{aligned} \Delta &\equiv a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4, \\ &\equiv b_1B_1 + b_2B_2 + b_3B_3 + b_4B_4, \\ &\equiv a_1A_1 + b_1B_1 + c_1C_1 + d_1D_1; \end{aligned}$$

and so on. Similarly for any other determinant.

500. ***If the elements in any row are multiplied by the co-factors of the corresponding elements in another row, the sum of the products vanishes.***

Thus, in the determinant $|a_1\ b_2\ c_3\ d_4|$,

$$b_1B_1 + b_2B_2 + b_3B_3 + b_4B_4 \equiv \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}.$$

No one of the co-factors B_1, B_2, B_3, B_4 contains any of the elements b_1, b_2, b_3, b_4. Hence, these co-factors are unaffected if in the above identity we change b_1, b_2, b_3, b_4 to a_1, a_2, a_3, a_4. This gives

$$a_1B_1 + a_2B_2 + a_3B_3 + a_4B_4 \equiv \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \equiv 0.$$

Similarly for any other case.

501. Evaluation of Determinants. By the use of § 491, § 493, and § 499 we can readily obtain the value of any numerical determinant.

Evaluate $$\begin{vmatrix} 3 & 1 & 4 & 1 \\ 1 & 3 & 2 & 1 \\ 2 & 1 & 3 & 3 \\ 4 & 3 & 2 & 3 \end{vmatrix}.$$

From the first row subtract 3 times the second, from the third twice the second, from the fourth 4 times the second. The result is

$$\begin{vmatrix} 0 & -8 & -2 & -2 \\ 1 & 3 & 2 & 1 \\ 0 & -5 & -1 & 1 \\ 0 & -9 & -6 & -1 \end{vmatrix},$$

which, by § 486, reduces to

$$-\begin{vmatrix} -8 & -2 & -2 \\ -5 & -1 & 1 \\ -9 & -6 & -1 \end{vmatrix} = \begin{vmatrix} 8 & 2 & 2 \\ 5 & 1 & -1 \\ 9 & 6 & 1 \end{vmatrix} = 70. \qquad (§ 485)$$

502. Simultaneous Equations. Consider the simultaneous equations

$$a_1x + b_1y + c_1z = k_1,$$
$$a_2x + b_2y + c_2z = k_2,$$
$$a_3x + b_3y + c_3z = k_3.$$

Write the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, and let A_1, A_2, B_1, B_2, etc., be the co-factors in this determinant.

Multiply the first equation by A_1, the second by A_2, the third by A_3, and add.

Then $(a_1A_1 + a_2A_2 + a_3A_3)x = k_1A_1 + k_2A_2 + k_3A_3,$

since (§ 500) $\qquad b_1A_1 + b_2A_2 + b_3A_3 = 0,$

and $\qquad c_1A_1 + c_2A_2 + c_3A_3 = 0.$

Hence (§ 499),

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} x = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}, \quad \text{or} \quad x = \frac{|k_1\ b_2\ c_3|}{|a_1\ b_2\ c_3|}.$$

In a similar manner,

$$y = \frac{|a_1\ k_2\ c_3|}{|a_1\ b_2\ c_3|}; \qquad z = \frac{|a_1\ b_2\ k_3|}{|a_1\ b_2\ c_3|}.$$

Similarly for any set of simultaneous equations of the first degree.

503. Elimination. To eliminate x, y, and z from the four equations

$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0,$$
$$a_3x + b_3y + c_3z + d_3 = 0,$$
$$a_4x + b_4y + c_4z + d_4 = 0,$$

we substitute in the fourth equation the values of x, y, z found from the first three; *viz.* (§ 502),

$$x = -\frac{|d_1\, b_2\, c_3|}{|a_1\, b_2\, c_3|};\quad y = -\frac{|a_1\, d_2\, c_3|}{|a_1\, b_2\, c_3|};\quad z = -\frac{|a_1\, b_2\, d_3|}{|a_1\, b_2\, c_3|}.$$

Then $$-a_4\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} - b_4\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$- c_4\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} + d_4\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0,$$

or $$-a_4|b_1\, c_2\, d_3| + b_4|a_1\, c_2\, d_3| - c_4|a_1\, b_2\, d_3| + d_4|a_1\, b_2\, c_3| = 0.$$

This equation, by § 499, may be written

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0.$$

Observe that this determinant is the determinant formed by the sixteen coefficients.

Similarly for any other set of simultaneous equations.

This determinant is called the **eliminant** of the system of the given equations.

The eliminant of a system of n equations with $n-1$ unknowns is the determinant formed by eliminating all the unknowns from the system.

(1) Eliminate y and z from the equations

$$\begin{aligned} 2x^2 + 3y + z &= 0, \\ 3x + 1 + y + 2z &= 0, \\ 4x^2 - 3y + 4z &= 0. \end{aligned}$$

The result is
$$\begin{vmatrix} 2x^2 & 3 & 1 \\ 3x+1 & 1 & 2 \\ 4x^2 & -3 & 4 \end{vmatrix} = 0,$$
which reduces to
$$8x^2 - 9x - 3 = 0.$$

(2) Eliminate x from the two equations

$$4x^2 + 3xy + 5 = 0, \qquad [1]$$
$$2y^2 + 3x + 4 = 0. \qquad [2]$$

$$\left.\begin{array}{lllll} \text{[1] is} & 4x^2 + & 3yx & + \quad 5 & = 0 \\ x \times [2], & 3x^2 + (2y^2+4)x & & & = 0 \\ \text{Transpose [2],} & & 3x & + (2y^2+4) & = 0 \end{array}\right\}.$$

Represent x^2 by u, and eliminate u and x.

Then,
$$\begin{vmatrix} 4 & 3y & 5 \\ 3 & 2y^2+4 & 0 \\ 0 & 3 & 2y^2+4 \end{vmatrix} = 0.$$

(3) Eliminate x from the two equations

$$ax^2 + bx + c = 0, \qquad [1]$$
$$a'x + c' = 0. \qquad [2]$$

$$\left.\begin{array}{ll} \text{[1] is} & ax^2 + bx + c = 0 \\ \text{Multiply [2] by } x, & a'x^2 + c'x \quad = 0 \\ \text{[2] is} & \qquad a'x + c' = 0 \end{array}\right\}.$$

Eliminate x^2 and x,
$$\begin{vmatrix} a & b & c \\ a' & c' & 0 \\ 0 & a' & c' \end{vmatrix} = 0,$$
which reduces to
$$\frac{a}{a'^2} + \frac{c}{c'^2} - \frac{b}{a'c'} = 0.$$

This must be the condition that there exists a value of x which satisfies both equations, since it is assumed that such is the case when we apply the process of elimination.

We have obtained, therefore, the condition that the two given equations have a common root.

In general, the eliminant of a system of n equations with $n-1$ unknowns is that function of the coefficients of the equations which becomes zero when the equations have common roots, and only then.

Exercise 71

1. In the determinant $|a_1\ b_2\ c_3\ d_4|$ write the co-factors of a_3, b_2, b_4, c_1, c_4, d_2, d_3.

2. Express as a single determinant

$$\begin{vmatrix} e & f & g \\ f & h & k \\ g & k & l \end{vmatrix} + \begin{vmatrix} b & e & g \\ c & f & k \\ d & g & l \end{vmatrix} + \begin{vmatrix} b & g & f \\ c & k & h \\ d & l & k \end{vmatrix} + \begin{vmatrix} b & f & e \\ c & h & f \\ d & k & g \end{vmatrix}.$$

3. Write all the terms of the following determinant which contain a:

$$\begin{vmatrix} a & 0 & b & c & b \\ a & b & c & b & 0 \\ 0 & c & b & c & 0 \\ 0 & 0 & 0 & b & c \\ b & 0 & 0 & c & b \end{vmatrix}.$$

Expand:

4. $\begin{vmatrix} a & b & b & a \\ b & a & a & b \\ a & a & b & b \\ 0 & a & b & b \end{vmatrix}$. **5.** $\begin{vmatrix} 0 & d & d & d \\ a & 0 & a & a \\ b & b & 0 & b \\ c & c & c & 0 \end{vmatrix}$. **6.** $\begin{vmatrix} 1 & a & a & a \\ 1 & b & a & a \\ 1 & a & b & a \\ 1 & a & a & b \end{vmatrix}$.

Find the value of:

7. $\begin{vmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{vmatrix}$. **8.** $\begin{vmatrix} 3 & 2 & 1 & 4 \\ 15 & 29 & 2 & 14 \\ 16 & 19 & 3 & 17 \\ 33 & 39 & 8 & 38 \end{vmatrix}$. **9.** $\begin{vmatrix} 2 & 1 & 3 & 4 \\ 7 & 4 & 5 & 9 \\ 3 & 3 & 6 & 2 \\ 1 & 7 & 7 & 5 \end{vmatrix}$.

Solve the equations:

10. $\left.\begin{aligned} 3x - 4y + 2z &= 1 \\ 2x + 3y - 3z &= -1 \\ 5x - 5y + 4z &= 7 \end{aligned}\right\}.$ **11.** $\left.\begin{aligned} 4x - 7y + z &= 16 \\ 3x + y - 2z &= 10 \\ 5x - 6y - 3z &= 10 \end{aligned}\right\}.$

12. $$\left.\begin{aligned} 4x+7y+3z-3w &= 6 \\ 2x-y-4z+3w &= 13 \\ 3x+2y-7z-4w &= 2 \\ 5x-3y+z+5w &= 13 \end{aligned}\right\}.$$

13. $$\left.\begin{aligned} 3x+2y+4z-w &= 13 \\ 5x+y-z+2w &= 9 \\ 2x+3y-7z+3w &= 14 \\ 4x-4y+3z-5w &= 4 \end{aligned}\right\}.$$

14. Eliminate y from the equations

$$\left.\begin{aligned} x^2+2xy+3x+4y+1 &= 0 \\ 4x+3y+1 &= 0 \end{aligned}\right\}.$$

15. Eliminate m from the equations

$$\left.\begin{aligned} m^2x-2mx^2+1 &= 0 \\ m+x^2-3mx &= 0 \end{aligned}\right\}.$$

Find the eliminant of:

16. $$\left.\begin{aligned} ax^2+bx+c &= 0 \\ x^3 &= 1 \end{aligned}\right\}.$$

17. $$\left.\begin{aligned} ax^2+bx+c &= 0 \\ a'x^2+b'x+c' &= 0 \end{aligned}\right\}.$$

18. $$\left.\begin{aligned} ax^2+bx+c &= 0 \\ x^3+qx+r &= 0 \end{aligned}\right\}.$$

Are the following equations consistent?

19. $$\left.\begin{aligned} 4x^2+3x+2 &= 0 \\ 2x^2+x+1 &= 0 \end{aligned}\right\}.$$

20. $$\left.\begin{aligned} 3x^2+4xy+4x+1 &= 0 \\ x-3y-7 &= 0 \\ 2x-y-4 &= 0 \end{aligned}\right\}.$$

21. If ω is one of the complex cube roots of 1, show that:

$$\begin{vmatrix} 1 & -\omega & \omega^2 \\ -\omega & \omega^2 & 1 \\ \omega^2 & 1 & -\omega \end{vmatrix} = -4; \quad \begin{vmatrix} 1 & \omega & \omega^2 & 1 \\ \omega & \omega^2 & 1 & 1 \\ \omega^2 & 1 & 1 & \omega \\ 1 & 1 & \omega & \omega^2 \end{vmatrix} = 3\sqrt{-3}.$$

22. Show that in any determinant there are two terms which have all but two elements alike; and that these two terms have different signs.

23. Show that the sign of a determinant of order $4m + 2$ or $4m + 3$ is unchanged if the order of both columns and rows is reversed.

504. Product of Two Determinants. Consider the determinant

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 \\ a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 \\ a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}.$$

By § 491 this determinant may be expressed as the sum of twenty-seven determinants, of which the following are types:

$$\begin{vmatrix} a_1\alpha_1 & a_2\alpha_1 & a_3\alpha_1 \\ a_1\alpha_2 & a_2\alpha_2 & a_3\alpha_2 \\ a_1\alpha_3 & a_2\alpha_3 & a_3\alpha_3 \end{vmatrix}, \quad \begin{vmatrix} a_1\alpha_1 & a_2\alpha_1 & b_3\beta_1 \\ a_1\alpha_2 & a_2\alpha_2 & b_3\beta_2 \\ a_1\alpha_3 & a_2\alpha_3 & b_3\beta_3 \end{vmatrix}, \quad \begin{vmatrix} a_1\alpha_1 & b_2\beta_1 & c_3\gamma_1 \\ a_1\alpha_2 & b_2\beta_2 & c_3\gamma_2 \\ a_1\alpha_3 & b_2\beta_3 & c_3\gamma_3 \end{vmatrix}.$$

There are three determinants of the first type, eighteen of the second type, and six of the third type. Those of the first and second types are easily seen to vanish (§§ 489, 490). There remain the six determinants of the third type.

Consider any one of these six determinants, as

$$\begin{vmatrix} c_1\gamma_1 & a_2\alpha_1 & b_3\beta_1 \\ c_1\gamma_2 & a_2\alpha_2 & b_3\beta_2 \\ c_1\gamma_3 & a_2\alpha_3 & b_3\beta_3 \end{vmatrix}.$$

This may be written

$$c_1a_2b_3\begin{vmatrix} \gamma_1 & \alpha_1 & \beta_1 \\ \gamma_2 & \alpha_2 & \beta_2 \\ \gamma_3 & \alpha_3 & \beta_3 \end{vmatrix}, \quad \text{or} \quad -c_1a_2b_3\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

It is evident that the number of interchanges required to bring the columns into the order $\alpha\ \beta\ \gamma$ is the same as the number of inversions among the letters α, β, γ; and also the

same as the number of inversions among the letters a, b, c. Hence, the sign is + if that number is even, and − if that number is odd. The sign before $c_1a_2b_3$ is, therefore, the sign of this term in the determinant $|a_1\ b_2\ c_3|$ (§ 482, II).

Since the preceding is true for each of the six determinants of the third type, the given determinant is the product of the determinant $|a_1\ b_2\ c_3|$ by the determinant $|\alpha_1\ \beta_2\ \gamma_3|$, and is one of the forms in which the product

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

may be written.

The above proof is perfectly general and may be extended to the product of any two determinants.

(1) Write as a determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \times \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

The result is $\begin{vmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{vmatrix}$.

(2) Write as a determinant the product

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \times \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}.$$

The result is $\begin{vmatrix} X & Y & Z \\ Z & X & Y \\ Y & Z & X \end{vmatrix}$,

where $X = ax + by + cz$, $Y = cx + ay + bz$, $Z = bx + cy + az$.

505. The notation $\left\Vert \begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{matrix} \right\Vert = 0$

is used to denote that the four determinants obtained by omitting in turn one of the four columns all vanish.

Exercise 72

1. Show that $\begin{vmatrix} a & b & 0 \\ c & 0 & c \\ 0 & b & a \end{vmatrix} \times \begin{vmatrix} 0 & a & b \\ c & 0 & c \\ b & a & 0 \end{vmatrix} \equiv -4a^2b^2c^2.$

2. Express as a single determinant

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}.$$

3. Express as a single determinant

$$\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} \times \begin{vmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix},$$

and thence resolve the first determinant into its simplest factors.

4. Express as a single determinant

$$\begin{vmatrix} a+bi & -c+di \\ c+di & a-bi \end{vmatrix} \times \begin{vmatrix} \alpha+\beta i & -\gamma+\delta i \\ \gamma+\delta i & \alpha-\beta i \end{vmatrix},$$

where $i = \sqrt{-1}$; and thence prove Euler's theorem, *viz.*, *the product of two sums of four squares can itself be expressed as the sum of four squares.*

5. Show that $\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2.$

6. Show that $\begin{vmatrix} b+c & c+a & a+b \\ b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \end{vmatrix} \equiv 2 \begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}.$

CHAPTER XXX

GENERAL PROPERTIES OF EQUATIONS

506. Algebraic Functions. A function of a variable x has already been defined (§ 373) as any expression that changes in value when x changes in value. Any expression that involves x is, in general, a function of x. If x is involved only in a finite number of powers and roots, the expression is an **algebraic function** of x.

Thus, x^3, $\sqrt{x^3+x}$, $\dfrac{1}{x^3+4}$ are algebraic functions of x; but a^x, $\log x$, are not algebraic functions of x.

507. Rational Integral Functions. An algebraic function of x is **rational** as regards x, if x is involved only in powers; that is, not in roots. An algebraic function of x is **rational** and **integral** as regards x, if x is involved only in positive integral powers; that is, in numerators and not in denominators.

Thus, $\dfrac{1}{x^2}$, x^{-3}, $\dfrac{1}{4x+3}$, $\dfrac{x}{x^2+a^2}$, $\dfrac{3x^2+4}{5x^3+3x+2}$ are rational, but not integral, algebraic functions of x; while $4x^2+3x+7$, ax^3+bx^2+cx+d are rational integral algebraic functions of x.

508. Quantics. An algebraic function that is rational and integral with regard to all the variables in it is called a **quantic**.

We shall consider in this chapter only functions of one variable, and by quantic will be meant a rational integral algebraic function of one variable, unless it is expressly stated that several variables are involved.

NOTE. The term *quantic* is generally applied only to homogeneous expressions like $ax^2+bxy+cy^2$. This expression is obtained from ax^2+bx+c by putting $\dfrac{x}{y}$ for x, and multiplying through by y^2. The

theory of the two expressions is precisely the same, and we shall therefore extend the term *quantic* to include expressions like $ax^2 + bx + c$, $ax^3 + bx^2 + cx + d$, etc.

The **degree** of a quantic that involves only one variable x is indicated by the exponent of the highest power of x involved in the quantic (§ 122).

A quantic of the first degree is called a *linear function*; quantics of higher degrees are called *quadratics*, *cubics*, *biquadratics* or *quartics*, *quintics*, etc.

509. General Form. Any quantic of the nth degree in which x is the variable may be written in the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are coefficients which do not involve x. Some of these coefficients may be zero, and in that case the corresponding terms are wanting.

The *coefficients* may be real or complex, surd or rational expressions. We shall, in general, consider only quantics that have real and rational coefficients. The student will readily see what properties of such quantics are possessed by quantics that have surd or complex coefficients.

510. Abbreviations. For brevity a quantic that involves x is often represented by $f(x)$, $F(x)$, $\phi(x)$, or some similar notation.

If any quantic is represented by $f(x)$, it is represented by $f(a)$ when a is put for x.

Thus, if $\qquad f(x) \equiv 2x^3 - x^2 + 3x + 4,$

$$f(2) = 2(2)^3 - 2^2 + 3(2) + 4 = 16 - 4 + 6 + 4 = 22.$$

511. Equations. Every equation that contains no variables except rational integral algebraic functions of x can, by the transposition of all the terms to the first member, be made to assume the form $f(x) = 0$, where $f(x)$ is a quantic that involves the one variable x. The theory of this quantic and

that of the corresponding equation are closely related, and we shall develop the two together.

The **roots** of the equation $f(x)=0$ are those values of x that satisfy the equation. These roots are also called the *roots of the quantic.*

The **degree** of the equation $f(x)=0$ is the same as that of the quantic $f(x)$.

512. Divisibility of Quantics. Theorem I. *If* h *is a root of the equation* $\mathrm{f(x)}=0$, *the quantic* $\mathrm{f(x)}$ *is divisible by* $\mathrm{x-h}$.

For example, consider the quantic

$$f(x)\equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n. \qquad [1]$$

Now, since h is a root of the equation $f(x)=0$, we have

$$0=a_0h^n+a_1h^{n-1}+a_2h^{n-2}+\cdots+a_{n-1}h+a_n. \qquad [2]$$

Subtract [2] from [1],

$$a_0(x^n-h^n)+a_1(x^{n-1}-h^{n-1})+a_2(x^{n-2}-h^{n-2})+\cdots$$
$$+a_{n-2}(x^2-h^2)+a_{n-1}(x-h).$$

Each of the expressions x^n-h^n, $x^{n-1}-h^{n-1}$, $x^{n-2}-h^{n-2}$, $\cdots$, $x-h$, is divisible by $x-h$ (§ 86). Therefore, $f(x)$ is divisible by $x-h$. Similarly for any other quantic. Compare §§ 87, 495.

513. Theorem II. *Conversely, if a quantic* $\mathrm{f(x)}$ *is divisible by* $\mathrm{x-h}$, *then* h *is a root of the equation* $\mathrm{f(x)}=0$.

For, if $\phi(x)$ is the quotient obtained by dividing $f(x)$ by $x-h$, we have

$$f(x)\equiv(x-h)\,\phi(x).$$

Hence, equation $f(x)=0$ may be written

$$(x-h)\,\phi(x)=0,$$

of which h is evidently a root (§ 124).

514. Synthetic Division. Divide the quantic

$$3x^5-4x^4+x^3-12x^2+3x+6 \text{ by } x-2.$$

$$\begin{array}{llll}
3x^5-4x^4+x^3-12x^2+3x+6 & \underline{|\,x-2\qquad\qquad\qquad\qquad} \\
\underline{3x^5-6x^4} & 3x^4+2x^3+5x^2-2x-1 \\
\quad +2x^4+x^3 & \\
\quad \underline{+2x^4-4x^3} & \\
\qquad\quad +5x^3-12x^2 & \\
\qquad\quad \underline{+5x^3-10x^2} & \\
\qquad\qquad\quad -2x^2+3x & \\
\qquad\qquad\quad \underline{-2x^2+4x} & \\
\qquad\qquad\qquad\quad -x+6 & \\
\qquad\qquad\qquad\quad \underline{-x+2} & \\
\qquad\qquad\qquad\qquad\quad +4 &
\end{array}$$

The work may be abridged as follows, by omitting the powers of x and writing only the coefficients (§ 70):

$$\begin{array}{ll}
3-4+1-12+3+6 & \underline{|\,1-2\qquad\qquad\quad} \\
\underline{3-6} & 3+2+5-2-1 \\
\quad +2+1 & \\
\quad \underline{+2-4} & \\
\qquad +5-12 & \\
\qquad \underline{+5-10} & \\
\qquad\quad -2+3 & \\
\qquad\quad \underline{-2+4} & \\
\qquad\qquad -1+6 & \\
\qquad\qquad \underline{-1+2} & \\
\qquad\qquad\quad +4 &
\end{array}$$

The operation may be still further abridged. As the first term of the divisor is unity, the first term of each remainder is the next term of the quotient, and we need not write the quotient. Further, we need not bring down the several terms of the dividend or write the first terms of the partial products. Thus,

$$\begin{array}{l}
3-4+1-12+3+6\,\underline{|\,1-2} \\
\quad \underline{-6} \\
\quad +2 \\
\qquad \underline{-4} \\
\qquad +5 \\
\qquad\quad \underline{-10} \\
\qquad\quad -2 \\
\qquad\qquad \underline{+4} \\
\qquad\qquad -1 \\
\qquad\qquad\quad \underline{+2} \\
\qquad\qquad\quad +4
\end{array}$$

If we omit the first term of the divisor, which is now useless, change -2 to $+2$, and add, we may shorten the work to

$$\begin{array}{r|l} 3-4+1-12+3+6 & 2 \\ +6+4+10-4-2 & \\ \hline 3+2+5-\ \ 2-1+4 & \end{array}$$

The last term below the line is the remainder, the preceding terms the coefficients of the quotient. In this particular problem the quotient is $3x^4+2x^3+5x^2-2x-1$, and the remainder is 4.

This method is called **synthetic division.** For dividing any quantic in x by $x-h$ we have the following rule:

Arrange the quantic according to descending powers of x, *supplying any missing powers of* x *by these powers with zero coefficients.*

Write the coefficients a, b, c, *etc., in a horizontal line.*

Bring down the first coefficient a.

Multiply a *by* h, *and add the product to* b.

Multiply this sum by h, *and add the product to* c.

Continue this process; the last sum is the remainder, and the preceding sums the coefficients of the quotient.

Divide $2x^4-6x^2+5x-2$ by $x-3$.

$$\begin{array}{r|l} 2+0-\ \ 6+\ \ 5-\ \ \ 2 & 3 \\ +6+18+36+123 & \\ \hline 2+6+12+41+121 & \end{array}$$

The quotient is $2x^3+6x^2+12x+41$, and the remainder 121.

515. Value of a Quantic. If we put h for x in the quantic,

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

$$f(h) \equiv a_0h^n + a_1h^{n-1} + \cdots + h_{n-1}x + a_n.$$

$$\therefore f(x)-f(h) \equiv a_0(x^n-h^n)+a_1(x^{n-1}-h^{n-1})$$
$$+\cdots+a_{n-1}(x-h). \qquad [1]$$

Divide the right member of [1] by $x-h$ and represent the quotient by $\phi(x)$.

Then, $\qquad f(x)-f(h) \equiv (x-h)\,\phi(x).$

$$\therefore f(x) \equiv (x-h)\,\phi(x)+f(h).$$

Hence, the value which a quantic f(x) *assumes when we put* h *for* x *is equal to the last remainder obtained in the operation of dividing* f(x) *by* x − h.

This remainder, and, consequently, the value of the quantic, may be easily calculated by synthetic division.

The truth of the above theorem may also be shown by another method, which has the advantage of showing the form of the quotient and remainder.

For example, divide the quantic $ax^4 + bx^3 + cx^2 + dx + e$ by $x - h$.

$$\begin{array}{ccccc|l} a & b & c & d & e & h \\ & ah & Bh & Ch & Dh & \\ \hline a & B & C & D & R & \end{array}$$

where
$$\begin{aligned} B &= ah + b, \\ C &= Bh + c = ah^2 + bh + c, \\ D &= Ch + d = ah^3 + bh^2 + ch + d, \\ R &= Dh + e = ah^4 + bh^3 + ch^2 + dh + e. \end{aligned}$$

The remainder R is evidently the value which the quantic assumes when we put h for x.

The quotient is

$$ax^3 + (ah + b)x^2 + (ah^2 + bh + c)x + (ah^3 + bh^2 + ch + d).$$

Similarly for any other quantic.

Exercise 73

Find the quotient and the remainder obtained by dividing each of the following quantics by the divisor opposite it:

1. $x^4 - 3x^3 - x^2 + 2x - 1.$ $\qquad x - 2.$
2. $x^4 - 3x^2 + 2x - 7.$ $\qquad x - 3.$
3. $2x^4 + 3x^3 - 8x^2 - 7x - 10.$ $\qquad x - 2.$
4. $3x^4 + 2x^3 - 6x + 50.$ $\qquad x + 3.$
5. $ax^3 + 3bx^2 + 3cx + d.$ $\qquad x + h.$

Determine whether each of the following numbers is a root of the quantic opposite it (§ 513):

6. (3). $x^4 + x^2 - 6x + 2 = 0.$

7. (− 7). $x^4 + 7x^3 + 21x + 147 = 0.$

8. (0.3). $x^4 - 2.3x^3 + 3.6x^2 + 4.9x + 1.2 = 0.$

Find the value of each of the following quantics when for x we put the number opposite:

9. $3x^3 + 2x^2 - 6x + 1.$ (− 3).

10. $2x^4 + 6x^3 - 9x - 5.$ (6).

11. $x^5 + 7x^3 - 2x^2 - 49.$ (− 4).

12. $x^4 + 6x^3 - 7x^2 - 3x + 1.$ (− 0.2).

516. Number of Roots. We shall assume that every rational integral equation has at least one root. The proof of this truth is beyond the scope of the present chapter.*

Let $f(x)$ be a rational integral quantic of the nth degree, and let $f(x) = 0$. This equation has, by assumption, at least one root. Let α_1 be a root.

Then, by § 512, $\quad f(x) \equiv (x - \alpha_1) f_1(x),$

where $f_1(x)$ is a quantic of degree $n - 1$.

The equation $f_1(x) = 0$ must, by assumption, have a root. Let α_2 be a root.

Then, by § 512, $\quad f_1(x) \equiv (x - \alpha_2) f_2(x),$

where $f_2(x)$ is a quantic of degree $n - 2$.

Continuing this process, we see that at each step the degree of the quotient is diminished by one. Hence, we can find n factors $x - \alpha_1, x - \alpha_2, \cdots, x - \alpha_n$. The last quotient will not involve x, and is readily seen to be a_0, the coefficient of x^n in $f(x)$.

* See Burnside and Panton's *Theory of Equations*. H. Weber's *Traité d'Algèbre Supérieure*.

Now,
$$\begin{aligned} f(x) &\equiv (x-\alpha_1)f_1(x) \\ &\equiv (x-\alpha_1)(x-\alpha_2)f_2(x) \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\equiv a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n). \end{aligned}$$

Therefore, the equation $f(x)=0$ may be written

$$a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)=0,$$

which evidently holds true if x has any one of the n values $\alpha_1, \alpha_2, \cdots, \alpha_n$.

It follows, then, that if *every* rational integral equation has at least one root, *an equation of the* n*th degree has* n *roots.*

517. Linear Factors. The factors $x-\alpha_1, x-\alpha_2, \cdots, x-\alpha_n$ are linear functions of x (§ 508).

When $f(x)$ is written in the form

$$a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n),$$

it is said to be *resolved into its linear factors.*

From § 516 it follows that a quantic can be resolved into linear factors in only one way.

To resolve a quantic $f(x)$ into its linear factors is evidently equivalent to solving the equation $f(x)=0$.

518. Multiple Roots. The n roots of an equation of the nth degree are not necessarily all different.

The equation $x^3-7x^2+15x-9=0$ may be written

$$(x-1)(x-3)(x-3)=0,$$

and the roots are seen to be 1, 3, 3.

The root 3 and the corresponding factor $x-3$ occur twice; hence, 3 is said to be a *double root.* When a root occurs three times it is called a *triple root;* four times, a *quadruple root;* and so on.

Any root that occurs more than once is a **multiple root.**

519. Roots given. When all the roots of an equation are given the equation can at once be written.

Write the equation of which the roots are 1, 2, 4, -5.

The equation is $\quad (x-1)(x-2)(x-4)(x+5)=0,$

or $\quad x^4 - 2x^3 - 21x^2 + 62x - 40 = 0.$

520. Solutions by Trial. When all the roots but two of an equation can be found by trial the equation can be readily solved by the process of § 516. The work can be much abbreviated by employing the method of synthetic division (§ 514). (Compare § 180.)

Solve the equation $x^4 - 3x^3 - 6x^2 + 14x + 12 = 0.$

Try $+1$ and -1. Substituting these values for x, we obtain

$$1 - 3 - 6 + 14 + 12 = 0,$$
$$1 + 3 - 6 - 14 + 12 = 0,$$

which are both false, so that neither $+1$ nor -1 is a root.

Try $+2$. Dividing by $x-2$,

$$\begin{array}{rrrrr|l} 1 & -3 & -6 & +14 & +12 & 2 \\ & +2 & -2 & -16 & -4 & \\ \hline 1 & -1 & -8 & -2 & +8 & \end{array}$$

we see that $+2$ is not a root.

Try -2. Dividing by $x+2$,

$$\begin{array}{rrrrr|l} 1 & -3 & -6 & +14 & +12 & -2 \\ & -2 & +10 & -8 & -12 & \\ \hline 1 & -5 & +4 & +6 & 0 & \end{array}$$

we see that -2 is a root. The quotient is $x^3 - 5x^2 + 4x + 6$.

In this quotient try -2 again. Dividing by $x+2$,

$$\begin{array}{rrrr|l} 1 & -5 & +4 & +6 & -2 \\ & -2 & +14 & -36 & \\ \hline 1 & -7 & +18 & -30 & \end{array}$$

we see that -2 is not again a root.

Try $+3$. Dividing by $x-3$,

$$\begin{array}{rrrr|l} 1 & -5 & +4 & +6 & 3 \\ & +3 & -6 & -6 & \\ \hline 1 & -2 & -2 & 0 & \end{array}$$

we see that $+3$ is a root. The quotient is $x^2 - 2x - 2$.

Hence, the given equation may be written

$$(x+2)(x-3)(x^2-2x-2)=0.$$

Therefore, one of the three factors must vanish.

If $x+2=0$, $x=-2$; if $x-3=0$, $x=3$; if $x^2-2x-2=0$, by solving this quadratic, we find $x=1+\sqrt{3}$ or $x=1-\sqrt{3}$.

Hence, the four roots of the given equation are

$$-2,\quad 3,\quad 1+\sqrt{3},\quad 1-\sqrt{3}.$$

Exercise 74

Solve:

1. $x^3-7x^2+16x-12=0.$
2. $x^3-5x^2-2x+24=0.$
3. $x^3-6x^2+6x+99=0.$
4. $x^3+9x^2+2x-48=0.$
5. $x^3-4x^2-8x+8=0.$
6. $x^3+2x^2+4x+3=0.$
7. $6x^3-29x^2+14x+24=0.$
8. $2x^3+3x^2-13x-12=0.$
9. $x^4-15x^2-10x+24=0.$
10. $x^4+5x^3-5x^2-45x-36=0.$
11. $x^4+4x^3-29x^2-156x+180=0.$
12. $x^4-5x^3-2x^2+12x+8=0.$
13. $6x^4-5x^3-30x^2+20x+24=0.$
14. $4x^4+8x^3-23x^2-7x+78=0.$

Form the equation which has for its roots:

15. $2, 6, -7.$
16. $2, 4, -3.$
17. $2, 0, -2.$
18. $2, 1, -2, -1.$
19. $0.2, 0.125, -0.4.$
20. $5, 3+\sqrt{-1}, 3-\sqrt{-1}.$
21. $2, \frac{1}{2}, 2, -\frac{1}{2}.$
22. $2, 3, -2, -3, -6.$
23. $\frac{1}{3}, \frac{2}{3}, -\frac{1}{2}, -\frac{3}{2}.$
24. $0.3, -0.2, -\frac{1}{20}, -\frac{6}{5}.$
25. $3+\sqrt{2}, 3-\sqrt{2}, 2+\sqrt{3}, 2-\sqrt{3}.$
26. $2+\sqrt{-1}, 2-\sqrt{-1}, 1+2\sqrt{-1}, 1-2\sqrt{-1}.$

521. Relations between the Roots and the Coefficients. The quadratic equation of which the roots are α and β is (§ 193)

$$(x-\alpha)(x-\beta)=0,$$

or
$$x^2-(\alpha+\beta)x+\alpha\beta=0.$$

The cubic equation of which the roots are α, β, γ is

$$(x-\alpha)(x-\beta)(x-\gamma)=0,$$

or
$$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\alpha\gamma+\beta\gamma)x-\alpha\beta\gamma=0.$$

The biquadratic equation of which the roots are α, β, γ, δ is

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)=0,$$

or
$$x^4-(\alpha+\beta+\gamma+\delta)x^3+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x^2$$
$$-(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)x+\alpha\beta\gamma\delta=0.$$

Similarly for equations of higher degree.

Take any equation in which the highest power of x has the coefficient unity. From the above we have the following relations between the roots and the coefficients:

The coefficient of the *second* term, with its sign changed, is the sum of the roots.

The coefficient of the *third* term is the sum of all the products that can be formed by taking the roots *two* at a time.

The coefficient of the *fourth* term, with its sign changed, is the sum of all the products that can be formed by taking the roots *three* at a time.

The coefficient of the *fifth* term is the sum of all the products that can be formed by taking the roots *four* at a time; and so on.

If the number of roots is *even*, the last term is the product of all the roots. If the number of roots is *odd*, the last term, *with its sign changed*, is the product of all the roots.

Observe that the sign of the coefficient is changed when an *odd* number of roots is taken to form a product; that the sign is unchanged when an *even* number of roots is taken to form a product.

522. Reduction to the p Form. By dividing the equation by the coefficient of the highest power of x, any rational integral algebraic equation whatever can be reduced to a form in which the coefficient of the highest power of x is unity.

We shall write an equation reduced to this form, called the *p form*, as follows:

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0.$$

Let α, β, γ, $\cdots$ be the roots of this equation. Represent by $\Sigma\alpha$ the sum of the roots, by $\Sigma\alpha\beta$ the sum of all the products that can be formed by taking the roots two at a time; and so on (§ 102).

From § 521 we now have

$$\begin{aligned} \Sigma\alpha &= -p_1, & p_1 &= -\Sigma\alpha, \\ \Sigma\alpha\beta &= +p_2, & p_2 &= +\Sigma\alpha\beta, \\ \Sigma\alpha\beta\gamma &= -p_3, & p_3 &= -\Sigma\alpha\beta\gamma, \\ &\cdots & &\cdots \\ \alpha\beta\gamma\delta\cdots &= (-1)^n p_n. & p_n &= (-1)^n\alpha\beta\gamma\delta\cdots \end{aligned}$$

Thus, let α, β, γ be the roots of the equation

$$x^3 - 7x^2 - 9x + 4 = 0.$$

Then,

$$\begin{aligned} \Sigma\alpha &= \alpha + \beta + \gamma = 7, \\ \Sigma\alpha\beta &= \beta\gamma + \gamma\alpha + \alpha\beta = -9, \\ \alpha\beta\gamma &= -4. \end{aligned}$$

The relations between the roots and the coefficients of an equation do not assist us to solve the equation. In every case we are brought at last to the original equation.

Thus, in the equation

$$x^3 - 7x^2 - 9x + 4 = 0,$$

we have

$$\begin{aligned} \alpha + \beta + \gamma &= 7, \\ \beta\gamma + \gamma\alpha + \alpha\beta &= -9, \\ \alpha\beta\gamma &= -4. \end{aligned}$$

If we eliminate any two of the three unknowns as β and γ, we have to solve the equation

$$\alpha^3 - 7\alpha^2 - 9\alpha + 4 = 0.$$

That is, we have to solve the given equation.

523. Symmetric Functions of the Roots. The expressions $\Sigma\alpha$, $\Sigma\alpha\beta$, $\Sigma\alpha\beta\gamma$, $\cdots$ are examples of **symmetric functions** of the roots (§ 192). Any expression that involves all the roots, and all the roots have the same exponents and the same coefficients, is a symmetric function of the roots.

From the relations

$$\Sigma\alpha = -p_1, \quad \Sigma\alpha\beta = +p_2, \quad \Sigma\alpha\beta\gamma = -p_3, \quad \cdots$$

the value of any symmetric function of the roots of a given equation may be found in terms of the coefficients.

If α, β, γ are the roots of the equation

$$x^3 - 4x^2 + 6x - 5 = 0,$$

we may calculate the values of symmetric functions of the roots as follows:

We have
$$\alpha + \beta + \gamma = 4, \qquad [1]$$
$$\beta\gamma + \gamma\alpha + \alpha\beta = 6, \qquad [2]$$
$$\alpha\beta\gamma = 5. \qquad [3]$$

(1) $\Sigma\alpha^2 \equiv \alpha^2 + \beta^2 + \gamma^2$.

Square [1], $\quad \alpha^2 + \beta^2 + \gamma^2 + 2\beta\gamma + 2\gamma\alpha + 2\alpha\beta = 16$

Multiply [2] by 2, $\quad 2\beta\gamma + 2\gamma\alpha + 2\alpha\beta = 12$

$$\therefore \alpha^2 + \beta^2 + \gamma^2 = 4 \qquad [4]$$

(2) $\Sigma\alpha^2\beta \equiv \alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta$.

Multiply [1] by [2], $\quad \Sigma\alpha^2\beta + 3\alpha\beta\gamma = 24$

Multiply [3] by 3, $\quad 3\alpha\beta\gamma = 15$

$$\therefore \Sigma\alpha^2\beta = 9 \qquad [5]$$

(3) $\Sigma\alpha^3 \equiv \alpha^3 + \beta^3 + \gamma^3$.

Multiply [1] by [4], $\quad \alpha^3 + \beta^3 + \gamma^3 + \Sigma\alpha^2\beta = 16$

[5] is $\quad \Sigma\alpha^2\beta = 9$

$$\therefore \alpha^3 + \beta^3 + \gamma^3 = 7$$

Similarly for any symmetric function of the roots. (Compare § 192.)

524. By the aid of the preceding sections we can find the condition that a given relation should exist among the roots of an equation.

Find the condition that the roots of the equation

$$x^3 + px^2 + qx + r = 0$$

shall be in geometrical progression.

Let β be the mean root. Then,

$$\alpha + \beta + \gamma = -p, \qquad [1]$$

$$\beta\gamma + \gamma\alpha + \alpha\beta = q, \qquad [2]$$

$$\alpha\beta\gamma = -r, \qquad [3]$$

and $$\beta^2 = \gamma\alpha. \qquad [4]$$

From [2] and [4], $$\beta\gamma + \alpha\beta + \beta^2 = q,$$

or, $$\beta(\gamma + \alpha + \beta) = q.$$

By [1], $$-p\beta = q.$$

$$\therefore \beta = -\frac{q}{p}.$$

Substitute in [3], β^2 for $\gamma\alpha$ and $-\frac{q}{p}$ for β,

$$\left(-\frac{q}{p}\right)^3 = -r.$$

$$\therefore q^3 = p^3r, \text{ the required condition.}$$

525. Complex Roots. If a complex number is a root of an equation with real coefficients, the conjugate complex number (§ 216) is also a root.

Let $\alpha + \beta i$, where $i = \sqrt{-1}$, be a root of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0,$$

the coefficients of which are real.

Put $\alpha + \beta i$ for x in the left member of this equation, and expand the powers of $\alpha + \beta i$ by the binomial theorem. All the terms which do not contain i, and all the terms which contain even powers of i, are real; all the terms which contain odd powers of i are orthotomic. Representing the real part of the result by P, and the orthotomic part of the result by Qi, we have (§ 511), since $\alpha + \beta i$ is a root,

$$P + Qi = 0.$$

Therefore, $$P = 0 \text{ and } Q = 0. \qquad (§ 219)$$

Now put $\alpha - \beta i$ for x in the given equation. The result may be obtained from the former result by changing i to $-i$. The even powers of i are unchanged, while the odd powers have their signs changed. The real part, therefore, is unchanged, and the orthotomic part is changed only in sign. The result is

$$P - Qi = 0,$$

which vanishes, since by the preceding $P = 0$ and $Q = 0$.

Therefore, $\alpha - \beta i$ is a root of the given equation (§ 511).

This theorem is generally stated as follows:

Complex roots always enter an equation in pairs.

Corresponding to a pair of complex roots, we shall have the factors

$$x - \alpha - \beta i,\ x - \alpha + \beta i.$$

The product of these,

$$(x - \alpha)^2 + \beta^2,$$

is positive, provided x is real. Hence, corresponding to a pair of complex roots, we have a factor of the second degree, which for real values of x does not change sign (§ 220).

Exercise 75

1. Form the equations of which the roots are

$$2,\ 4,\ -3; \qquad 3,\ -2,\ -4.$$

If α, β, γ are the roots of $x^3 - 5x^2 + 4x - 3 = 0$, find the value of:

2. $\Sigma\alpha^2$.	5. $\Sigma\alpha^2\beta\gamma$.	8. $\Sigma\alpha^4$.
3. $\Sigma\alpha^2\beta$.	6. $\Sigma\alpha^2\beta^2$.	9. $\Sigma\alpha^3\beta\gamma$.
4. $\Sigma\alpha^3$.	7. $\Sigma\alpha^3\beta$.	10. $\Sigma\alpha^2\beta^2\gamma$.

If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, find in terms of the coefficients the value of:

11. $\Sigma\alpha^2$.

12. $\Sigma\alpha^2\beta$.

13. $\Sigma\alpha^3$.

14. $\Sigma\alpha^2\beta^2$.

15. $\Sigma\alpha^4$.

16. $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$.

17. $\dfrac{\beta\gamma}{\alpha} + \dfrac{\gamma\alpha}{\beta} + \dfrac{\alpha\beta}{\gamma}$.

18. $\dfrac{\beta^2 + \gamma^2}{\beta\gamma} + \dfrac{\gamma^2 + \alpha^2}{\gamma\alpha} + \dfrac{\alpha^2 + \beta^2}{\alpha\beta}$.

19. $\dfrac{\beta^2 + \gamma^2}{\beta + \gamma} + \dfrac{\gamma^2 + \alpha^2}{\gamma + \alpha} + \dfrac{\alpha^2 + \beta^2}{\alpha + \beta}$.

In the equation $x^3 + px^2 + qx + r = 0$, find the condition:

20. That one root is the negative of another root.

21. That one root is double another.

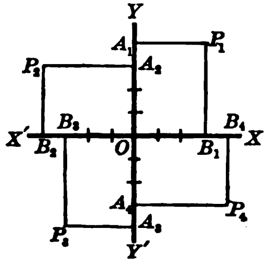
22. That the three roots are in arithmetical progression.

23. That the three roots are in harmonical progression.

GRAPHICAL REPRESENTATION OF FUNCTIONS

The investigation of the changes in the value of $f(x)$ corresponding to changes in the value of x is much facilitated by using the system of graphical representation explained in the following sections.

526. Coördinates. Let XX' be a horizontal line and let YY' be a line perpendicular to XX' at the point O.



Any point in the plane of the lines XX' and YY' is determined by its *distance* and *direction* from each of the perpendiculars XX' and YY'. Its distance from YY', measured on XX', is called the **abscissa** of the point; its distance from XX', measured on YY', is called the **ordinate** of the point.

Thus, the abscissa of P_1 is OB_1, the ordinate of P_1 is OA_1;
the abscissa of P_2 is OB_2, the ordinate of P_2 is OA_2;
the abscissa of P_3 is OB_3, the ordinate of P_3 is OA_3;
the abscissa of P_4 is OB_4, the ordinate of P_4 is OA_4.

The abscissa and the ordinate of a point are called the **coördinates** of the point; and the lines XX' and YY' are called the **axes of coördinates**. XX' is called the **axis of abscissas** or the **axis of x**; YY' is called the **axis of ordinates** or the **axis of y**; and the point O is called the **origin**.

An abscissa is generally represented by x, an ordinate by y.

The coördinates of a point are written thus: (x, y).

Thus, $(7, 4)$ is the point of which the abscissa is 7 and the ordinate 4.

Abscissas measured to the *right* of YY' are **positive**, to the *left* of YY' are **negative**. Ordinates measured *above* XX' are **positive**, *below* XX' are **negative**.

Thus, the points P_1, P_2, P_3, P_4 are respectively $(3, 4)$, $(-4, 3)$, $(-3, -4)$, $(4, -3)$.

527. It is evident that if a point is given, its coördinates referred to given axes may be easily found.

Conversely, if the coördinates of a point are given, the point may be readily constructed.

Thus, to construct the point $(4, -3)$, a convenient length is taken as a unit of length. A distance of 4 units is laid off on OX to the *right*, from O to B_4; and a distance of 3 units on OY' *downwards*, from O to A_4. The intersection of the perpendiculars erected at B_4 and A_4 determines the required point P_4.

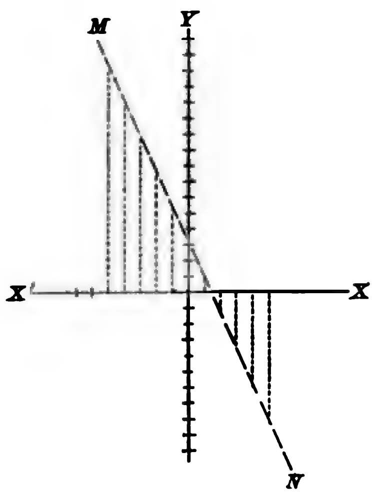
Construct the points $(3, 2)$; $(5, 4)$; $(6, -3)$; $(-4, -3)$; $(-4, 2)$; $(-3, -5)$; $(4, -3)$.

528. Graph of a Function. Let $f(x)$ be any function of x, where x is a variable. Put $y = f(x)$; then y is a new variable connected with x by the relation $y = f(x)$. If $f(x)$ is a rational integral function of x, it is evident that to every value of x there corresponds one, and only one, value of y.

If different values of x are laid off as abscissas, and the corresponding values of $f(x)$ as ordinates, the points thus obtained all lie on a line. This line in general is a curved line, or a *curve*, and is called the **graph of the function f(x)**; it is also called the **locus of the equation y = f(x)**.

(1) Construct the graph of $3 - 2x$.

Put $y = 3 - 2x$. The following table is readily computed:



If	$x = +1,$	$y = +1;$
	$x = +2,$	$y = -1;$
	$x = +3,$	$y = -3;$
	$x = +4,$	$y = -5;$
	$x = +5,$	$y = -7.$
If	$x = -1,$	$y = +5;$
	$x = -2,$	$y = +7;$
	$x = -3,$	$y = +9;$
	$x = -4,$	$y = +11;$
	$x = -5,$	$y = +13.$

Constructing the above points, it appears that the graph of the function $3 - 2x$ is the straight line MN.

In general, if the quantic $f(x)$ contains only the first powers of x and y, the graph is a straight line.

Exercise 76

Construct the graphs of the following functions:

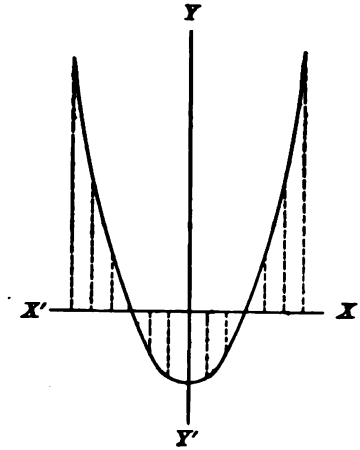
1. $3x + 2$.
2. $x - 5$.
3. $x + 6$.
4. $\frac{2}{3}(x - 5)$.
5. $\frac{1}{2}(x + 6)$.
6. $\frac{1}{3}(7 - 2x)$.
7. $\frac{3}{4}(9 - 3x)$.
8. $\frac{2}{3}(4 + 5x)$.
9. $(x - 2)(x - 3)$.
10. $5x^2 - 17x - 12$.

(2) Plot the graph of $\frac{1}{2}x^2 - 4$.

Put $y = \frac{1}{2}x^2 - 4$. We readily compute the following table:

If $x = +0$,	$y = -4$;
$x = \pm 1$,	$y = -3.5$;
$x = \pm 2$,	$y = -2$;
$x = \pm 3$,	$y = +0.5$;
$x = \pm 4$,	$y = +4$;
$x = \pm 5$,	$y = +8.5$;
$x = \pm 6$,	$y = +14$.

Plotting these points, we obtain the curve here given.



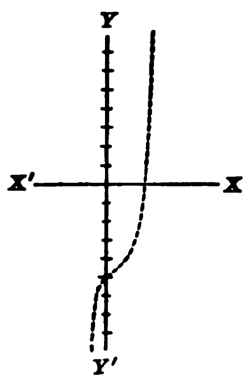
(3) Plot the graph of

$$x^3 - x^2 + x - 5.$$

Put $y = x^3 - x^2 + x - 5$. We compute the following table:

If $x = +0.5$,	$y = -\ 4.625$;
$x = +1.0$,	$y = -\ 4.000$;
$x = +1.5$,	$y = -\ 2.375$;
$x = +2.0$,	$y = +\ 1.000$;
$x = +2.5$,	$y = +\ 6.875$;
$x = +0.0$,	$y = -\ 5.000$;
$x = -0.5$,	$y = -\ 5.875$;
$x = -1.5$,	$y = -12.125$.

Interpolation (§ 433) shows that if $y = 0$, $x = 1.88+$. Does the result agree with the figure?

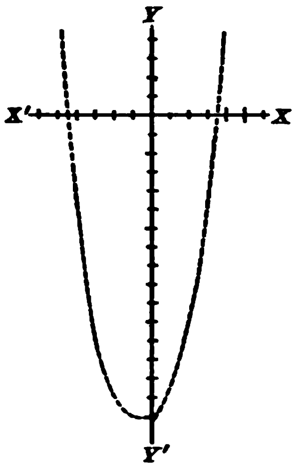


529. Consider any rational integral algebraic function of x, for example $x^2 + x - \frac{63}{4}$.

Put $$y = x^2 + x - \tfrac{63}{4}.$$

Assume values of x, compute the corresponding values of y, and construct the graph. Now, any value of x which makes

$y = 0$ satisfies the equation $x^2 + x - \frac{63}{4} = 0$, and is a root of that equation. Hence, any abscissa whose corresponding ordinate is zero represents a root of this equation. The roots may be found, approximately, by measuring the abscissas of the points in which the graph meets XX', for at these points $y = 0$.



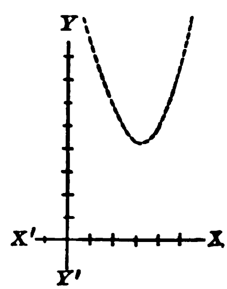
From the given equation the following table may be formed:

If	$x = +0$,	$y = -15.75$;
	$x = +1$,	$y = -13.75$;
	$x = +2$,	$y = -9.75$;
	$x = +3$,	$y = -3.75$;
	$x = +4$,	$y = +4.25$.
If	$x = -1$,	$y = -15.75$;
	$x = -2$,	$y = -13.75$;
	$x = -3$,	$y = -9.75$;
	$x = -4$,	$y = -3.75$;
	$x = -5$,	$y = +4.25$.

The table shows that one root of $f(x) = 0$ lies between 3 and 4 (since y changes from $-$ to $+$, and therefore passes through zero); and, for a like reason, the other root lies between -4 and -5.

530. An equation of any degree may be thus plotted, and the graph will be found to cross the axis XX' as many times as there are *real* roots in the equation.

When an equation has no real roots the graph does not meet XX'.



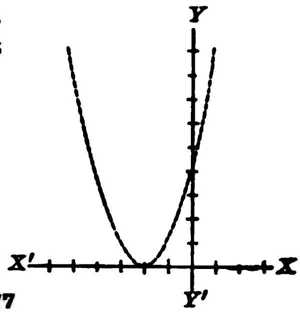
In the equation $x^2 - 6x + 13 = 0$, both of whose roots are imaginary, the graph, at its nearest approach, is 4 units distance from XX'.

If an equation has a double root, its graph touches XX', but does not intersect it at the point of contact.

The equation

$$x^2 + 4x + 4 = 0$$

has the roots -2 and -2, and the graph is as shown in the figure.



Exercise 77

Construct the graphs of the following functions:

1. $x^2 + 3x - 10$.
2. $x^3 - 2x^2 + 1$.
3. $x^4 - 20x^2 + 64$.
4. $x^2 - 4x + 10$.
5. $x^4 - 5x^2 + 4$.
6. $x^3 - 4x^2 + x - 1$.

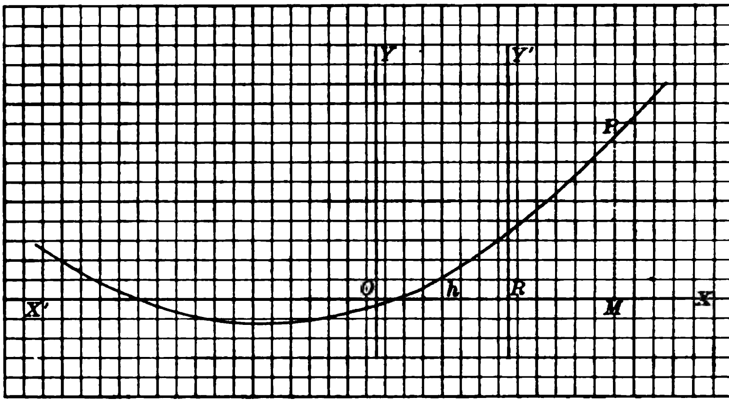
531. Change of Origin. Consider the function

$$y = x^2 + 4x - 1. \qquad [1]$$

Construct the graph of the given function, for convenience using on the coördinate paper 3 spaces for 1 horizontal unit, and 1 space for 4 vertical units.

If $x = +0$, $y = -1$; $\quad$ If $x = -1$, $y = -4$;
$x = +1$, $y = +4$; $\quad$ $x = -2$, $y = -5$;
$x = +2$, $y = +11$; $\quad$ $x = -3$, $y = -4$;
$x = +3$, $y = +20$; $\quad$ $x = -4$, $y = -1$;
$x = +4$, $y = +31$; $\quad$ $x = -5$, $y = +4$;
$x = +5$, $y = +44$. $\quad$ $x = -6$, $y = +11$.

Now change the origin from its present position O to any point R on the axis of abscissas, keeping the axis of ordinates RY' parallel to its original position OY. This change does not alter the values of the ordinates of points, but does alter the values of the abscissas.



The value of the given function of x *is altered.*

For, let $OR = h$. The old coördinates of any point P of the graph are $OM = x$, $MP = y$. Let x' denote the new abscissa RM of the point P.

Then, $$x = x' + h.$$

Substitute $x' + h$ for x in [1].

Then, $$y = (x' + h)^2 + 4(x' + h) - 1$$
$$= x'^2 + (4 + 2h)x' + h^2 + 4h - 1.$$

Write x for x', and we have the transformed function

$$y = x^2 + (4 + 2h)x + h^2 + 4h - 1.$$

Hence, when the origin is moved along the axis of x a distance of h units, the new function of x is obtained by substituting $x + h$ for x in the old function of x.

If the origin is moved a distance of h units to the *left* of O, the value of h must of course be regarded as *negative*.

Exercise 78

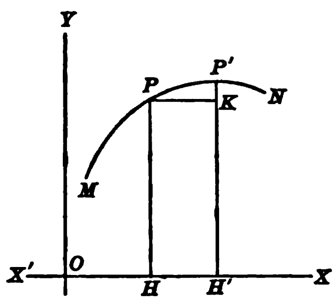
1. Transform the function $y = x^2 - 6x + 5$ to a new origin, the point (5, 0).

2. Transform the function $y = 4x^2 + 3x - 10$ to a new origin, the point $(-2, 0)$.

3. Transform the function $y = 3x^3 - 10x^2 + 9x - 2$ to a new origin, the point $(2, 0)$.

4. Transform the function $y = x^4 - 1$ to a new origin, the point $(-1, 0)$.

DERIVATIVES



532. Definitions. Let MN be a part of the graph of a function of x, as

$$f(x) = 2 + \sqrt{12x - x^2}.$$

Let $$y = 2 + \sqrt{12x - x^2}. \qquad [1]$$

Let P be any point on the graph. Draw the coördinates OH and HP of that point.

Let $$x \equiv OH, \text{ and } y \equiv HP.$$

It is obvious from [1] that $$y = f(x). \qquad [2]$$

Add to x any arbitrary amount HH'.
Draw $H'P' \perp$ to XX', and draw $PK \parallel$ to XX'.

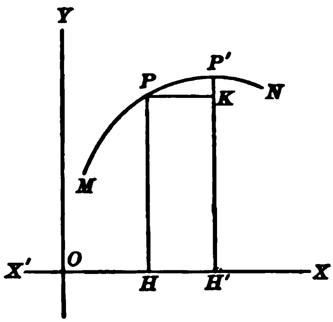
Let $$x' \equiv OH', \text{ and } y' \equiv P'H'.$$

It is obvious from [1] that

$$y' = f(x'). \qquad [3]$$

It is evident that when HH' is added to x, y changes to y', and that the amount of change in y is KP'.

The arbitrary amount HH' added to x is called the **increment of x.** This is written Δx and read delta x.



Similarly, the amount KP' added to y is called the **increment of y.**

Let $\Delta x \equiv$ the increment of x,

and $\quad \Delta y \equiv$ the increment of y.

Then, $\quad \Delta y = KP'$,

and since it is added to y, the increment is *positive.*

When the increment of y is taken from y, the increment is *negative.*

Hence, *an increment may be either positive or negative.*

The **increment of a variable** is any arbitrary amount added to the variable.

The **increment of a function** is the amount of the change produced in the function when an increment is given to the variable of the function.

Now, $\qquad x' = x + \Delta x.$

Hence, by [3], $\qquad y' = f(x + \Delta x). \qquad [4]$

Again, $\qquad \Delta y = y' - y.$

Hence, by [2] and [4], $\quad \Delta y = f(x + \Delta x) - f(x). \qquad [5]$

Therefore, to find the increment of a function when the variable takes an increment,

Subtract the original value of the function from the value of the function after the variable has taken an increment.

Divide [5] by Δx.

Then, $$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}. \qquad [6]$$

Now let P remain fixed and let P' move towards P along the curve in such a way that we can make it approach P as nearly as we please.

Then, Δx is an infinitesimal, and the fraction $\frac{\Delta y}{\Delta x}$ is, in general, a variable, and this variable, in general, approaches a definite limit.

When the variable does approach a definite limit this limit is called the **derivative of y** or the **derivative of $f(x)$**.

The derivative of $f(x)$ is $\underset{\Delta x \doteq 0}{\text{limit}} \frac{\Delta y}{\Delta x}$.

By [6] it is seen that the derivative of $f(x)$ is

$$\underset{\Delta x \doteq 0}{\text{limit}} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \qquad [7]$$

The first form of this definition is the more significant when we wish to show the relation of the increments to the derivative; the second is the more significant when we wish to show the relation of the function to the derivative.

The derivative with respect to x of $f(x)$ is represented by $D_x f(x)$; that of $f(y)$ with respect to y by $D_y f(y)$; that of v with respect to u by $D_u v$; and so on.

The derivative of $f(x)$ with respect to x is also represented by $f'(x)$.

Thus, $D_x f(x) \equiv f'(x)$; $D_y f(y) \equiv f'(y)$; and so on.

533. From [7] may be deduced the following rule for finding the derivative of a function:

Divide the increment of the function by the increment given to the variable.

Find the limit of this quotient when the increment of the variable is an infinitesimal.

This limit is the derivative of the function.

Denote the derivative of $f(x)$ by $f'(x)$.

Then, $$f'(x) \equiv \lim_{\Delta x \doteq 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

(1) Given $f(x)=5x^2$; find $f'(x)$.

$$f(x+\Delta x)=5(x+\Delta x)^2=5x^2+10x\Delta x+5(\Delta x)^2.$$
$$f(x)=5x^2.$$
$$f(x+\Delta x)-f(x)=10x\Delta x+5(\Delta x)^2.$$
$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=10x+5\Delta x.$$
$$\lim_{\Delta x \doteq 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}=10x.$$
$$\therefore f'(x)=10x.$$

(2) Find $D_x(x^3+4x+1)$.

The function is x^3+4x+1.

Change x to $x+h$, $(x+h)^3+4(x+h)+1$,

or $x^3+3hx^2+3h^2x+h^3+4x+4h+1$.

From the new value subtract the old,

$$3hx^2+3h^2x+h^3+4h.$$

Divide by h, $3x^2+3hx+h^2+4$.

Take the limit as h approaches 0 as a limit;

$$D_x(x^3+4x+1)=3x^2+4.$$

534. Derivative of x^n. The function is x^n. Changing x to $x+h$, we obtain $(x+h)^n$. Now $(x+h)^n$ can be expanded by the binomial theorem, and we obtain

$$(x+h)^n=x^n+nx^{n-1}h+\frac{n(n-1)}{1\cdot 2}x^{n-2}h^2+\cdots$$

From this new value of the function subtract x^n, the old value, and divide by h.

We now have

$$D_x(x^n) = \underset{h \doteq 0}{\text{limit}} \left[nx^{n-1} + \frac{n(n-1)}{1 \cdot 2} x^{n-2}h + \cdots \right] = nx^{n-1}.$$

The sum of the terms after the first approaches 0 as a limit by § 405. Hence,

To find the derivative with respect to x of any power of x,

Multiply by the exponent, and diminish the exponent of* x *by one.

Thus,
$$D_x(x^4) = 4x^3;\ D_x(x^{-3}) = -3x^{-4};$$
$$D_x \frac{1}{\sqrt{x^3}} = D_x(x^{-\frac{3}{2}}) = -\tfrac{3}{2} x^{-\frac{5}{2}}.$$

NOTE. It can be proved that this rule holds true whether n is integral or fractional, positive or negative.

The derivative of a constant is zero, since the increment of a constant is zero.

Exercise 79

Find the derivative with respect to x of:

1. x^2.
2. x^3.
3. $\frac{1}{x}$.
4. x^{-2}.
5. x^4.
6. $\frac{1}{x^3}$.
7. x^{-4}.
8. $x^2 + x$.
9. $x^3 + 2x^2$.
10. $(x + a)^2$.
11. $\frac{1}{x^2 - 3}$.
12. $(x + 1)^{-2}$.

535. Derivative of a Sum of Two or More Functions. Let $f(x)$ and $\phi(x)$ be two functions of x; their sum $f(x) + \phi(x)$ is also a function of x. Now,

$$\begin{aligned} D_x[f(x) + \phi(x)] &= \underset{h \doteq 0}{\text{limit}} \left[\frac{f(x+h) + \phi(x+h) - f(x) - \phi(x)}{h} \right] \\ &= \underset{h \doteq 0}{\text{limit}} \left[\frac{f(x+h) - f(x)}{h} \right] + \underset{h \doteq 0}{\text{limit}} \left[\frac{\phi(x+h) - \phi(x)}{h} \right] \\ &= D_x f(x) + D_x \phi(x). \end{aligned}$$

Similarly for the sum of any number of functions.

Hence, ***the derivative with respect to* x *of the sum of two or more functions of* x *is the sum of the derivatives with respect to* x *of the several functions.***

The above may be formulated,

$$D_x(f+\phi+\cdots)=D_xf+D_x\phi+\cdots$$

Here f is an abbreviation for $f(x)$, ϕ for $\phi(x)$, etc.

By means of the above and §§ 533, 534 the derivative with respect to x of any rational integral algebraic function of x may be found.

Find $D_x(2x^3+4x^2-8x+3)$.

$$\begin{aligned} D_x(2x^3+4x^2-8x+3) &= D_x(2x^3)+D_x(4x^2)-D_x(8x)+D_x(3)\\ &= 2D_xx^3+4D_xx^2-8D_xx+D_x3\\ &= 2(3x^2)+4(2x)-8(1)+0\\ &= 6x^2+8x-8. \end{aligned}$$

536. Derivative of a Product of Two or More Functions. Let $f(x)$ and $\phi(x)$ be two functions of x; their product $f(x)\phi(x)$ is a new function of x.

Now,

$$D_x[f(x)\phi(x)]=\underset{h\doteq 0}{\text{limit}}\left[\frac{f(x+h)\phi(x+h)-f(x)\phi(x)}{h}\right]$$

$$=\underset{h\doteq 0}{\text{limit}}\left[\frac{f(x+h)\phi(x+h)-f(x+h)\phi(x)+f(x+h)\phi(x)-f(x)\phi(x)}{h}\right]$$

$$=\underset{h\doteq 0}{\text{limit}}\left[f(x+h)\frac{\phi(x+h)-\phi(x)}{h}\right]+\underset{h\doteq 0}{\text{limit}}\left[\phi(x)\frac{f(x+h)-f(x)}{h}\right]$$

$$=f(x)D_x\phi(x)+\phi(x)D_xf(x),$$

since $$\underset{h\doteq 0}{\text{limit}}[f(x+h)]=f(x).$$

The above may be formulated

$$D_x(f\phi) = fD_x\phi + \phi D_x f.$$

Similarly for three or more functions. Thus,

$$D_x(f\phi F) = f\phi D_x F + fF D_x\phi + \phi F D_x f.$$

Hence, *the derivative with respect to* x *of the product of two or more functions of* x *is the sum of the several products found by multiplying the derivative with respect to* x *of each function by each one of the other functions.*

537. Derivative of $(x - a)^n$. (See note, page 441.)

$$D_x(x-a)^n = \underset{h \doteq 0}{\text{limit}}\left[\frac{(x-a+h)^n - (x-a)^n}{h}\right]$$

$$= \underset{h \doteq 0}{\text{limit}}\left[\frac{(x-a)^n + n(x-a)^{n-1}h + \cdots - (x-a)^n}{h}\right]$$

$$= \underset{h \doteq 0}{\text{limit}}\left[n(x-a)^{n-1} + \cdots\right] = n(x-a)^{n-1}.$$

Thus, $$D_x(x-3)^4 = 4(x-3)^3.$$

Exercise 80

Write the derivative with respect to x of:

1. $x^2 + 4.$
2. $x^3 + 3x^2 - 1.$
3. $x^4 + x^2 + 2.$
4. $x^5 - 3x^4 + x^3.$
5. $4x^4 + 6x^3 + 2.$
6. $6x^5 - 7x^2 + 7x.$
7. $3x^5 + 4x^4 + x^3 - x^2 - 6x + 5.$
8. $4x^5 - 2x^4 - x^3 + 6x^2 - 7.$
9. $(x-2)(x+3).$
10. $(x-1)(x-2)(x-3).$
11. $(x-3)^2(x+4).$
12. $(x-4)^2(x-2)(x+1).$
13. $(x-\alpha)^2(x-\beta)^2.$
14. $(x-\alpha)(x-\beta)(x-\gamma).$
15. $(x-2)(x-3)(x+5)(x+4).$
16. $(x^2+2)(x^2-4x+8).$

538. Successive Derivatives. The derivative of a function of x is, in general, a function of x and has, in general, a derivative with respect to x.

The derivative of the derivative is called the **second derivative**; the derivative of the second derivative, the **third derivative**; and so on.

By *derivative* is meant the first derivative, unless the contrary is expressly stated.

The second derivative with respect to x of $f(x)$ is represented by $D_x^2f(x)$, or by $f''(x)$; the third derivative by $D_x^3f(x)$, or by $f'''(x)$; and so on.

Evidently,
$$f''(x) = D_x^2f(x) = D_xD_xf(x);$$
$$f'''(x) = D_x^3f(x) = D_xD_x^2f(x) = D_xD_xD_xf(x);$$
and so on.

539. Values of the Derivatives. The value which $f(x)$ assumes when we put a for x is represented by $f(a)$.

Similarly, the value which $f'(x)$ assumes when we put a for x is represented by $f'(a)$; the value of $f''(x)$ by $f''(a)$; and so on.

Thus, if
$$f(x) \equiv x^3 - 2x^2 + x + 4,$$
then
$$f'(x) \equiv 3x^2 - 4x + 1,$$
$$f''(x) \equiv 6x - 4,$$
$$f'''(x) \equiv 6;$$
and $f^{iv}(x), f^{v}(x), \cdots$ all vanish.

If we put 2 for x, we obtain
$$f(2) = 6;\ f'(2) = 5;\ f''(2) = 8;\ f'''(2) = 6.$$
Similarly for any other function.

540. Sign of the Derivative. In the function $f(x)$ let x increase by the successive addition of very small increments. As x increases, the value of $f(x)$ will change, sometimes increasing, sometimes decreasing.

Suppose that x has reached a fixed value a; the corresponding values of $f(x)$ and $f'(x)$ are $f(a)$ and $f'(a)$.

By § 532, $f'(a) = \underset{h \doteq 0}{\text{limit}}\left[\dfrac{f(a+h) - f(a)}{h}\right].$

If $f(x)$ is *increasing* as x passes through the value a, $f(a+h) > f(a)$ and $f'(a)$ is *positive.*

If $f(x)$ is *decreasing* as x passes through the value a, $f(a+h) < f(a)$ and $f'(a)$ is *negative.*

Conversely, if $f'(a)$ is positive, $f(a+h) - f(a)$ is positive, and $f(x)$ is increasing as x passes through the value a.

If $f'(a)$ is negative, $f(a+h) - f(a)$ is negative, and $f(x)$ is decreasing as x passes through the value a.

Hence, *for a particular value of* x, *if* f'(x) *is positive,* f(x) *is increasing; and if* f'(x) *is negative,* f(x) *is decreasing; and conversely.*

Observe that we are speaking of increasing and decreasing *algebraically.*

Thus, if $f(x) \equiv x^3 - 3x^2 - 6x + 10,$

$f'(x) \equiv 3x^2 - 6x - 6.$

We find $f(1) = 2,\ f'(1) = -9.$

$\therefore f(x)$ is decreasing as x passes through the value 1; for example,

$f(1) = 2,\ f(1.1) = 1.101,$ and $1.101 < 2.$

Again, $f(3) = -8,\ f'(3) = +3.$

$\therefore f(x)$ is increasing as x passes through the value 3; for example,

$f(3) = -8,\ f(3.1) = -7.639,$ and $-7.639 > -8.$

Exercise 81

Write the successive derivatives with respect to x of:

1. $x^3 - 4x^2 + 2.$
2. $x^3 + 4x^2 - 5x.$
3. $2x^4 + 2x^3 - 4x + 1.$
4. $3x^4 + 3x^3 - x^2 + x.$
5. $4x^5 - 7x^3 + 5x - 2.$
6. $ax^3 + 3bx^2 + 3cx + d.$
7. $ax^4 + 4bx^3 + 6cx^2 + 4dx + e$
8. $(x - \alpha)^2(x - \beta).$
9. $(x - \alpha)(x - \beta)(x - \gamma).$
10. $(x - \alpha)^2(x - \beta)^2.$

Find whether each of the following functions is increasing or decreasing as x increases through the value set opposite:

11. $x^3 - x^2 + 1$. (2)
12. $x^4 - x^2 + 6x - 1$. (4)
13. $2x^4 + 3x^2 - 6x$. (1)
14. $4x^4 - 3x^2 + 4x - 6$. (−3)

541. Derivative in Terms of the Roots. Take the cubic

$$f(x) \equiv a(x-\alpha)(x-\beta)(x-\gamma).$$

Since $D_x(x-\alpha)=1$, $D_x(x-\beta)=1$, $D_x(x-\gamma)=1$ (§ 537), we have, by § 536,

$$f'(x) \equiv a(x-\beta)(x-\gamma) + a(x-\alpha)(x-\gamma) + a(x-\alpha)(x-\beta)$$
$$\equiv \frac{f(x)}{x-\alpha} + \frac{f(x)}{x-\beta} + \frac{f(x)}{x-\gamma}.$$

Similarly, for any quantic,

$$f'(x) \equiv \frac{f(x)}{x-\alpha_1} + \frac{f(x)}{x-\alpha_2} + \cdots + \frac{f(x)}{x-\alpha_n} \equiv \sum \frac{f(x)}{x-\alpha}.$$

542. Maxima and Minima. If, as x increases, $f(x)$ increases until x reaches a certain value a, but $f(x)$ begins to decrease as soon as x passes through the value a, the value $f(a)$ of $f(x)$, when $x = a$, is called a **maximum** value of $f(x)$.

If, as x increases, $f(x)$ decreases until x reaches a certain value a, but $f(x)$ begins to increase as soon as x passes through the value a, the value $f(a)$ of $f(x)$, when $x = a$, is called a **minimum** value of $f(x)$.

From these definitions and from § 541 it follows that for all *continuous* functions of x (see § 557), when $f(x)$ is a maximum or a minimum, $f'(x) = 0$; and conversely, in general, if $f'(x) = 0$, $f(x)$ is either a maximum or a minimum. In other words, the general condition for a maximum or a minimum value of $f(x)$ is $f'(x) = 0$.

Hence, the maxima and minima values of $f(x)$ are found by deriving $f'(x)$ from $f(x)$, and then solving the equation

$$f'(x) = 0.$$

For, let a denote a value of x which satisfies the equation $f'(x) = 0$. Then $f(a)$ is, in general, either a maximum or a minimum, and it may be determined by the algebraic sign of $f''(x)$ whether $f(a)$ is a maximum or a minimum.

Suppose that $f(a)$ is a maximum. Then $f(x)$ must be *increasing* just before $x = a$, and *decreasiny* just after $x = a$.

Therefore, $f'(x)$ must be *positive* just before $x = a$, and *negative* just after $x = a$ (§ 541). Hence, $f'(x)$ must be *decreasing* as it passes through the value 0 at the point for which $x = a$. Therefore, by § 541, $f''(x)$ must be *negative* when $x = a$; for $f''(x)$ has the same relation to $f'(x)$ that $f'(x)$ has to $f(x)$.

By similar reasoning it may be proved that if $f(a)$ is a minimum, $f''(x)$ must be *positive* when $x = a$.

Hence, $f(a)$ is a *maximum* when $f''(a)$ is *negative*, and $f(a)$ is a *minimum* when $f''(a)$ is *positive*.

The most important points to be determined in constructing a graph are the points which correspond to the maxima and minima values of the function in question.

Exercise 82

Find the maxima and the minima values of the following functions of x, and plot the graphs:

1. $y = x^2 - 6x + 7$.
2. $y = x^3 + 6x^2 - x - 30$.
3. $y = x^3 - 12x$.
4. $y = 4x^3 - 12x + 1$.
5. $y = x^4 + 4x^3 - 20x^2 + 4$.

543. Multiple Roots. In the quantic $f(x)$ let α be a triple root. Then, we can write (§ 518)

$$f(x) \equiv (x-\alpha)^3 \phi(x),$$

where the degree of $\phi(x)$ is less by 3 than that of $f(x)$.

By § 536, $f'(x) \equiv (x-\alpha)^3\phi'(x) + 3(x-\alpha)^2\phi(x)$

$$\equiv (x-\alpha)^2[(x-\alpha)\phi'(x) + 3\phi(x)].$$

Hence, if $f(x)$ has a triple root α, the factor $(x-\alpha)^2$ occurs in the H.C.F. of $f(x)$ and $f'(x)$.

Similarly for a multiple root of any order.

To find the multiple roots of $f(x)$,

Find the H.C.F. of f(x) *and* f′(x), *and resolve it into factors. Each root occurs once more in* f(x) *than the corresponding factor occurs in the H.C.F.*

Find the multiple roots of

$$x^5 - x^4 - 5x^3 + x^2 + 8x + 4 = 0.$$

Here, $\qquad f(x) \equiv x^5 - x^4 - 5x^3 + x^2 + 8x + 4,$

$$\therefore f'(x) \equiv 5x^4 - 4x^3 - 15x^2 + 2x + 8.$$

Find the H.C.F. of $f(x)$ and $f'(x)$ as follows:

$$\begin{array}{r|r|l}
5-4-15+2+8 & 5-5-25+5+40+20 & 1 \\
\underline{5+0-15-10} & \underline{5-4-15+2+8\qquad} & \\
-4+0+12+8 & -1-10+3+32+20 & \\
\underline{-4+0+12+8} & -5-50+15+160+100 & -1 \\
 & \underline{-5+4+15-2-8} & \\
 & -54)\,-54+0+162+108 & \\
 & 1-0-3-2 & -5+4
\end{array}$$

Hence, $x^3 - 3x - 2$ is the H.C.F.

We find, by substitution, that -1 is a root of the equation

$$x^3 - 3x - 2 = 0.$$

The other roots are found to be -1 and 2 (§ 520).

Hence, $\qquad x^3 - 3x - 2 \equiv (x+1)^2(x-2).$

Therefore, -1 is a triple root, and 2 is a double root, of the given equation. As the given equation is of the fifth degree, these are all the roots, and the equation may be written

$$(x+1)^3(x-2)^2 = 0.$$

Having found the multiple roots of an equation, we may divide by the corresponding factors and find the remaining roots, if any, from the reduced equation.

Exercise 83

The following equations have multiple roots. Find all the roots of each equation:

1. $x^3 - 8x^2 + 13x - 6 = 0.$

2. $x^3 - 7x^2 + 16x - 12 = 0.$

3. $x^4 - 6x^2 - 8x - 3 = 0.$

4. $x^4 - 7x^3 + 9x^2 + 27x - 54 = 0.$

5. $x^4 + 6x^3 + x^2 - 24x + 16 = 0.$

6. $x^5 - 11x^4 + 19x^3 + 115x^2 - 200x - 500 = 0.$

7. Resolve into linear factors

$$x^6 - 5x^5 + 5x^4 + 9x^3 - 14x^2 - 4x + 8.$$

8. Show that an equation of the form $x^n = a^n$ can have no multiple root.

9. Show that the condition that the equation

$$x^3 + 3qx + r = 0$$

shall have a double root is $4q^3 + r^2 = 0.$

10. Show that the condition that the equation

$$x^3 + 3px^2 + r = 0$$

shall have a double root is $r(4p^3 + r) = 0.$

544. Expansion of f(x + h). Consider a quantic of the fourth degree

$$f(x) \equiv ax^4 + bx^3 + cx^2 + dx + e.$$

Put $x + h$ in place of x. Then,

$$f(x+h) \equiv a(x+h)^4 + b(x+h)^3 + c(x+h)^2 + d(x+h) + e.$$

Expand the powers of $x + h$, and arrange the terms by descending powers of x.

$$\begin{array}{l|l|l|l|l}
f(x+h)\equiv a & x^4 + 4ah & x^3 + 6ah^2 & x^2 + 4ah^3 & x + ah^4\\
 & \quad + b & \quad + 3bh & \quad + 3bh^2 & \quad + bh^3\\
 & & \quad + c & \quad + 2ch & \quad + ch^2\\
 & & & \quad + d & \quad + dh\\
 & & & & \quad + e
\end{array}$$

But
$$\begin{aligned}
f(h) &= ah^4 + bh^3 + ch^2 + dh + e,\\
f'(h) &= 4ah^3 + 3bh^2 + 2ch + d,\\
f''(h) &= 12ah^2 + 6bh + 2c,\\
f'''(h) &= 24ah + 6b,\\
f^{\text{iv}}(h) &= 24a,\\
f^{\text{v}}(h) &= 0.
\end{aligned}$$

$$\therefore f(x+h)\equiv f(h) + xf'(h) + x^2\frac{f''(h)}{\underline{|2}} + x^3\frac{f'''(h)}{\underline{|3}} + x^4\frac{f^{\text{iv}}(h)}{\underline{|4}}.$$

If we arrange the expansion of $f(x + h)$ by ascending powers of h, we find

$$f(x+h)\equiv f(x) + hf'(x) + h^2\frac{f''(x)}{\underline{|2}} + h^3\frac{f'''(x)}{\underline{|3}} + h^4\frac{f^{\text{iv}}(x)}{\underline{|4}}.$$

Similarly for any other quantic.

545. Calculation of the Coefficients. The coefficients in the expansion of $f(x + h)$ may be conveniently calculated as follows:

Take $\qquad f(x) \equiv ax^4 + bx^3 + cx^2 + dx + e.$

Put $\qquad f(x + h) \equiv Ax^4 + Bx^3 + Cx^2 + Dx + E,$

where A, B, C, D, E are to be found.

In the last identity put $x - h$ for x.

Then, since $f(x - h + h) = f(x)$, we obtain

$$\begin{aligned}f(x) \equiv A(x-h)^4 &+ B(x-h)^3 + C(x-h)^2\\ &+ D(x-h) + E.\end{aligned}$$

From the last identity we derive the following rule for finding the coefficients of the powers of x in the expansion of $f(x+h)$.

Divide $f(x)$ by $x-h$; the remainder is E, that is, $f(h)$; and the quotient is

$$A(x-h)^3 + B(x-h)^2 + C(x-h) + D.$$

Divide this quotient by $(x-h)$; the remainder is D, that is, $f'(h)$; and the quotient is

$$A(x-h)^2 + B(x-h) + C.$$

Continue the division. The last quotient is A or a.

The above division is best arranged as follows (§ 515):

$$\begin{array}{ccccc|l}
a & b & c & d & e & \underline{h} \\
 & ah & b'h & c'h & d'h & \\
\hline
a & b' & c' & d' & E & \\
 & ah & b''h & c''h & & \\
\cline{1-4}
a & b'' & c'' & D & & \\
 & ah & b'''h & & & \\
\cline{1-3}
a & b''' & C & & & \\
 & ah & & & & \\
\cline{1-2}
a & B & & & &
\end{array}$$

Hence, $f(x+h) \equiv ax^4 + Bx^3 + Cx^2 + Dx + E.$

This method is easily extended to equations of any degree.

Exercise 84

In the following quantics put for x the expression opposite, and reduce:

1. $x^3 - 3x^2 + 4x - 6.$ $(x+2)$
2. $x^4 - 2x^2 + 6x - 3.$ $(x+4)$
3. $3x^4 - 2x^3 + 2x^2 - x - 4.$ $(x+3)$
4. $2x^4 - 3x^3 + 6x^2 - 7x - 8.$ $(x-2)$
5. $2x^4 - 2x^3 + 4x^2 - 5x - 4.$ $(x-3)$

TRANSFORMATION OF EQUATIONS

546. The solution of an equation and the investigation of its properties are often facilitated by a change in the form of the equation. Such a change of form is called a *transformation of the equation*.

547. Roots with Signs changed. *The roots of the equation* $f(-x) = 0$ *are those of the equation* $f(x) = 0$, *each with its sign changed.*

For, let α be any root of equation $f(x) = 0$.

Then we must have $f(\alpha) = 0$.

In the quantic $f(-x)$ put $-\alpha$ for x; that is, α for $-x$.

The result is $f(\alpha)$.

But we have just seen that $f(\alpha)$ vanishes, since α is a root of the equation $f(x) = 0$. Hence, $f(-x)$ vanishes when we put $-\alpha$ for x, and (§ 511) $-\alpha$ is, therefore, a root of the equation $f(-x) = 0$.

Similarly, the negative of each of the roots of $f(x) = 0$ is a root of $f(-x) = 0$; and, since the two equations are evidently of the same degree, these are all the roots of the equation

$$f(-x) = 0.$$

To obtain $f(-x)$ we change the sign of all the *odd* powers of x in the quantic $f(x)$.

Thus, the roots of the equation $x^4 - 2x^3 - 13x^2 + 14x + 24 = 0$ are 2, 4, -1, -3; and those of the equation $x^4 + 2x^3 - 13x^2 - 14x + 24 = 0$ are -2, -4, $+1$, $+3$.

548. Roots multiplied by a Given Number. Consider the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0. \qquad [1]$$

Put $y = mx$, then $x = \frac{y}{m}$. Then the equation becomes

$$a\left(\frac{y}{m}\right)^4 + b\left(\frac{y}{m}\right)^3 + c\left(\frac{y}{m}\right)^2 + d\left(\frac{y}{m}\right) + e = 0. \qquad [2]$$

The left member of [2] differs from the left member of [1] only in that $\frac{y}{m}$ is put in place of x.

Let α be any root of [1]; the left member of [1] vanishes when we put α for x.

That is, $$a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e = 0.$$

In the left member of [2] put $m\alpha$ for y.

Then, $$a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e,$$

which, as we have just seen, vanishes. Hence, if α is a root of [1], $m\alpha$ is a root of [2]. Since the above is true for each of the roots of [1], and the two equations are evidently of the same degree, the roots thus obtained are all the roots of [2].

Similarly for an equation of any degree.

Equation [2] may be written in the form

$$ay^4 + mby^3 + m^2cy^2 + m^3dy + m^4e = 0.$$

Hence, to write an equation the roots of which are the roots of a given equation multiplied by m,

Multiply the second term of the given equation by m; *the third term by* m^2; *and so on.*

Zero coefficients are to be supplied for missing powers of x.

Write the equation of which the roots are double the roots of the equation

$$3x^4 - 2x^3 + 4x^2 - 6x - 5 = 0.$$

Here $m = 2$, and the result is

$$3x^4 - 2(2)x^3 + 4(2)^2x^2 - 6(2)^3x - 5(2)^4 = 0,$$

or $$3x^4 - 4x^3 + 16x^2 - 48x - 80 = 0.$$

549. Removal of Fractional Coefficients. If any of the coefficients of an equation in the form

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0$$

are fractions, we can remove fractions as follows:

Multiply the roots by* m; *then take* m *so that all of the coefficients will be integers.

Reduce to an equation, in the p form, with integral coefficients $2x^3 - \frac{1}{3}x^2 + \frac{5}{6}x + \frac{1}{4} = 0$.

Divide by 2, $\qquad x^3 - \frac{1}{6}x^2 + \frac{5}{12}x + \frac{1}{8} = 0.$

Multiply the roots by m (§ 548),

$$x^3 - \frac{m}{6}x^2 + \frac{5m^2}{12}x + \frac{m^3}{8} = 0.$$

The least value of m that will render the coefficients all integral is seen to be 6. Put 6 for m, $x^3 - x^2 + 15x + 27 = 0$, the equation required.

550. Reciprocal Roots. Consider the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0. \qquad [1]$$

Put $y = \frac{1}{x}$; then $x = \frac{1}{y}$; and the equation becomes

$$a\left(\frac{1}{y}\right)^4 + b\left(\frac{1}{y}\right)^3 + c\left(\frac{1}{y}\right)^2 + d\left(\frac{1}{y}\right) + e = 0. \qquad [2]$$

Let α be any root of [1].

Then, $\qquad a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e = 0.$

In the left member of [2] put α for $\frac{1}{y}$; that is, $\frac{1}{\alpha}$ for y,

$$a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e,$$

which, as we have just seen, vanishes.

Hence, $\frac{1}{\alpha}$ is a root of [2]. Since the above is true for each root of [1], and the two equations are of the same degree, the reciprocals of the roots of [1] are all the roots of [2].

Similarly for an equation of any degree.

Equation [2] may be written

$$a + by + cy^2 + dy^3 + ey^4 = 0,$$

or, writing x in place of y,

$$ex^4 + dx^3 + cx^2 + bx + a = 0;$$

so that the coefficients are those of the given equation in reversed order.

Write the equation of which the roots are the reciprocals of the roots of $2x^4 - 3x^3 + 4x^2 - 5x - 7 = 0$.

The result is $\qquad 2 - 3x + 4x^2 - 5x^3 - 7x^4 = 0,$

or $\qquad 7x^4 + 5x^3 - 4x^2 + 3x - 2 = 0.$

551. Reciprocal Equations. The coefficients of an equation may be such that reversing their order does not change the equation. In this case the reciprocal of a root is another root of the equation. That is, half the roots are reciprocals of the other half. Such an equation is a **reciprocal equation.**

Thus, the roots of the equation

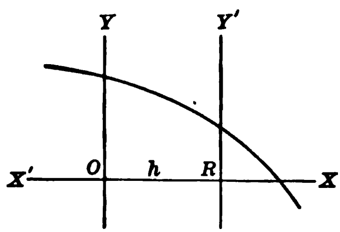
$$6x^5 - 29x^4 + 27x^3 + 27x^2 - 29x + 6 = 0$$

are $-1, 2, 3, \frac{1}{2}, \frac{1}{3}$. Here -1 is the reciprocal of itself; $\frac{1}{2}$ of 2; $\frac{1}{3}$ of 3.

552. Roots diminished by a Given Number. Consider the equation

$$f(x) = x^4 + ax^3 + bx^2 + cx + d = 0, \qquad [1]$$

and the corresponding graph with the point O as origin.



To diminish the roots of this equation by any number h is equivalent to changing the origin from the point O to a point R on the axis of x such that $OR = h$. The change is made (§ 531) by writing $x + h$ for x in [1]. The result is

$$f(x+h) \equiv (x+h)^4 + a(x+h)^3 + b(x+h)^2 + c(x+h) + d = 0. \qquad [2]$$

Denote the new coefficients of the equation by a_1, b_1, c_1, d_1.

Then, $\quad f(x+h) \equiv x^4 + a_1x^3 + b_1x^2 + c_1x + d_1 = 0.$ [3]

To find the values of a_1, b_1, c_1, and d_1, transpose the origin back to O by writing in [3] $x - h$ for x.

Then,

$$f(x) \equiv (x-h)^4 + a_1(x-h)^3 + b_1(x-h)^2 + c_1(x-h) + d_1 = 0. \quad [4]$$

Take out the factor $x - h$, and denote the quotient by Q.

Then, $\quad f(x) = (x - h)Q + d_1.$ [5]

Hence, d_1 is the remainder obtained by dividing $f(x)$ by $x - h$. Similarly, c_1 is the remainder obtained by dividing Q by $x - h$; and so on.

Therefore, the new coefficients are easily found by the repeated application of synthetic division to the coefficients of the given equation.

Evidently the same method may be applied to an equation of any degree.

To *increase the roots* by a given number h we have only to diminish the roots by the number $-h$.

Obtain the equation the roots of which are each less by 2 than the roots of the equation

$$2x^4 - 3x^3 - 4x^2 + 2x + 9 = 0.$$

The work (§ 545) is as follows:

2	−3	−4	+2	+9	2
	+4	+2	−4	−4	
2	+1	−2	−2	+5	
	+4	+10	+16		
2	+5	+8	+14		
	+4	+18			
2	+9	+26			
	+4				
2	+13				

The required equation is

$$2x^4 + 13x^3 + 26x^2 + 14x + 5 = 0.$$

553. Transformation in General. In the general problem of transformation we have given an equation in x, as $f(x)=0$, and we have to form a new equation in y where y is a given function of x, such as $\phi(x)$.

When from the equation $y=\phi(x)$ we can find a value of x, the transformation is made by substituting this value of x in the given equation, and reducing the result.

(1) Given the equation $x^3-3x+1=0$; to find the equation in y where $y=3x-2$.

We find $x=\dfrac{y+2}{3}$. Substitute in the given equation,

and we have $$\left(\frac{y+2}{3}\right)^3-3\left(\frac{y+2}{3}\right)+1=0,$$

which reduces to $$y^3+6y^2-15y-19=0.$$

(2) Given the equation $x^3-2x^2+3x-5=0$, of which α, β, γ are the roots; find the equation of which the roots are $\beta+\gamma-\alpha$, $\gamma+\alpha-\beta$, $\alpha+\beta-\gamma$.

We have $$y=\beta+\gamma-\alpha=\alpha+\beta+\gamma-2\alpha=2-2\alpha. \qquad (\S\ 521)$$

$$\therefore\ \alpha=\frac{2-y}{2}.$$

But, since α is a root of the given equation,

$$\alpha^3-2\alpha^2+3\alpha-5=0.$$

Put $\dfrac{2-y}{2}$ for α, and reduce.

Then, $y^3-2y^2+8y+24=0$, the equation required.

Exercise 85

Write the equations whose roots are the roots of the following equations multiplied by the number opposite:

1. $x^3-3x^2+2x-4=0.$ $\qquad(-1)$
2. $x^4+3x^3-2x-1=0.$ $\qquad(-2)$
3. $2x^4-3x^3+x^2-6x-4=0.$ $\qquad(-3)$

4. $2x^4 - 3x^2 + 6x - 8 = 0.$ (-2)

5. $3x^5 - 4x^3 - 2x + 7 = 0.$ (-2)

Transform to equations with integral coefficients in the p form:

6. $12x^3 - 4x^2 + 6x + 1 = 0.$

7. $6x^3 + 10x^2 - 7x + 16 = 0.$

8. $10x^4 + 5x^3 - 4x^2 + 25x - 30 = 0.$

9. $6x^5 + 3x^4 + 4x^3 - 2x^2 + 6x - 18 = 0.$

Write the equations which have for their roots the reciprocals of the roots of:

10. $3x^4 - 2x^3 + 5x^2 - 6x + 7 = 0.$

11. $2x^5 - 4x^3 - 5x^2 - 7x - 8 = 0.$

12. $x^6 - x^4 + 2x^2 + 4x - 1 = 0.$

Write the equations whose roots are the roots of the following equations diminished by the number opposite:

13. $x^3 - 11x^2 + 31x - 12 = 0.$ (1)

14. $x^4 - 6x^3 + 4x^2 + 18x - 5 = 0.$ (2)

15. $x^3 + 10x^2 + 13x - 24 = 0.$ (-2)

16. $x^4 + x^3 - 16x^2 - 4x + 48 = 0.$ (4)

17. $x^4 + x^2 - 3x + 4 = 0.$ (0.3)

18. $x^4 - 3x^3 - x^2 + 4x - 5 = 0.$ (-0.4)

19. $x^3 - 9x^2 + 22x - 12 = 0.$ (3)

20. Form the equation which has for its roots the squares of the roots of the equation $x^3 - 2x^2 + 3x - 5 = 0.$

21. Form the equation which has for its roots the squares of the differences of the roots of $x^3 - 4x^2 + 2x - 3 = 0.$

SITUATION OF THE ROOTS

554. Finite Value of a Quantic. Any positive integral power of x is finite as long as x is finite.

The product of a positive integral power of x by a finite number will be finite when x is finite.

A quantic consists of the sum of a definite number of such products, and has, consequently, a finite value as long as x is finite.

The *derivatives* of a quantic are new quantics and have, consequently, finite values as long as x is finite.

555. Sign of a Quantic. *When* x *is taken numerically large enough the sign of a quantic is the same as the sign of its first term.*

Write the quantic

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

in the form $$a_0x^n\left(1 + \frac{a_1}{a_0x} + \frac{a_2}{a_0x^2} + \cdots + \frac{a_n}{a_0x^n}\right).$$

By taking x large enough, each of the terms in parenthesis after the first can be made as small as we please.

If a_k is numerically the greatest of the coefficients $a_1, a_2, \cdots, a_n$, the sum of the terms in parenthesis after the first is numerically less than

$$\frac{a_k}{a_0}\left(\frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{x^n}\right);$$

that is (§ 280), less than $\frac{a_k}{a_0}\left(\dfrac{1 - \dfrac{1}{x^n}}{x - 1}\right)$.

The value of this expression can be made less than 1, or, indeed, less than *any* assigned value, by taking x large enough.

Hence, even in the most unfavorable case, that in which all the terms in parenthesis after the first are negative, the sum of these terms can still be made less than 1; the sum of all the terms in parenthesis is then positive. The sign of the quantic is the same as the sign of a_0x^n, its first term.

556. *When* x *is taken numerically small enough the sign of a quantic is the same as the sign of its last term.*

Write the quantic in the form

$$a_n\left(\frac{a_0x^n}{a_n}+\frac{a_1x^{n-1}}{a_n}+\cdots+\frac{a_{n-1}x}{a_n}+1\right).$$

The proof follows the proof of the last section.

557. Continuity of a Quantic. A function of x, $f(x)$, is **continuous** when an infinitesimal (§ 376) change in x always produces an infinitesimal change in $f(x)$, *whatever the value of* x.

We proceed to show that if $f(x)$ is a quantic in x, it is a continuous function of x.

Give to x *any* particular finite value a; the corresponding value of $f(x)$ is $f(a)$.

Increase x to $a+h$; the corresponding value of $f(x)$ is $f(a+h)$, and the increment in the value of $f(x)$ is

$$f(a+h)-f(a),$$

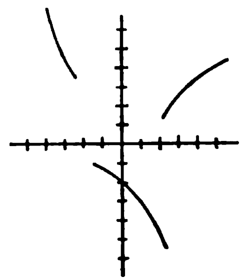
or

$$h\left(f'(a)+\frac{h}{\lfloor 2}f''(a)+\cdots+\frac{h^{n-1}}{\lfloor n}f^n(a)\right). \qquad (§\ 544)$$

The derivatives $f'(a)$, $f''(a)$, $\cdots$, $f^n(a)$ all have finite values (§ 554); and it is easily seen from § 556 that when h is very small the expression in parenthesis is numerically less than $2f'(a)$. Since $2hf'(a)$ approaches 0 as a limit (§ 379, I) when h approaches 0 as a limit, the increment of $f(x)$, which is less than $2hf'(a)$, approaches 0 as a limit when h approaches 0 as a limit.

Since the above is true for *any particular finite* value of x, we see that an infinitesimal change in x always produces an infinitesimal change in $f(x)$.

It follows that as $f(x)$ gradually changes from $f(a)$ to $f(b)$, it must pass through all intermediate values.



The derivatives of a quantic c in x are themselves quantics in x and are, therefore, continuous.

The changes in the value of a quantic $f(x)$ are well illustrated by the graph of the function. Since $f(x)$ is continuous, we can never have a graph in which there are *breaks* in the curve, as in the curve here given. In this curve there are breaks, or *discontinuities*, at $x = -2$ and $x = +2$.

558. Theorem on Change of Sign. *Let two real numbers* a *and* b *be put for* x *in* f(x). *If the resulting values of* f(x) *have contrary signs, an odd number of roots of the equation* f(x) = 0 *lie between* a *and* b.

As x changes from a to b, passing through all intermediate values, $f(x)$ will change from $f(a)$ to $f(b)$, passing through all intermediate values. Now in changing from $f(a)$ to $f(b)$, $f(x)$ changes sign.

Hence, $f(x)$ must pass through the value zero. That is, there is some value of x between a and b which causes $f(x)$ to vanish; that is, some root of the equation $f(x) = 0$ lies between a and b.

But $f(x)$ may pass through zero more than once. To change sign, $f(x)$ must pass through zero an *odd* number of times; and an odd number of roots must lie between a and b.

Applied to the graph of the equation, since to a root corresponds a point in which the graph meets the axis of x (§ 529), the above simply means that to pass from a point below the

axis of x to a point above that axis, we must cross the axis an odd number of times.

Thus, in $x^3 - 2x^2 + 3x - 7 = 0$, if we put 2 for x, the value of the left member is -1; if we put 3 for x, the value is $+11$. Hence, certainly one root lies between 2 and 3, and possibly all three roots of the equation lie between 2 and 3.

559. *An equation of odd degree has at least one real root the sign of which is opposite to that of the constant term.*

For, if the first coefficient is not positive, change signs so as to make it positive. If the last term is negative, make x positive and very large; the sign of the left member is $+$ (§ 555). Put $x = 0$; the sign of the left member is $-$. Hence, there is at least one real positive root.

Similarly, if the last term is positive, there is at least one real negative root.

560. Descartes' Rule of Signs. An equation in which all the powers of x from x^0 to x^n are present is said to be **complete**; if any powers of x are missing, the equation is said to be **incomplete.** An incomplete equation can be made complete by writing the missing powers of x with zero coefficients.

A **permanence** of sign occurs when $+$ follows $+$, or $-$ follows $-$; a **variation** of sign when $-$ follows $+$, or $+$ follows $-$.

Thus, in the complete equation

$$x^6 - 3x^5 + 2x^4 + x^3 - 2x^2 - x - 3 = 0,$$

writing only the signs

$$+ \quad - \quad + \quad + \quad - \quad - \quad -,$$

we see that there are three *variations* of sign and three *permanences.*

For *positive* roots, Descartes' rule is as follows:

The number of positive roots of the equation $f(x) = 0$ *cannot exceed the number of variations of sign in the quantic* $f(x)$.

To prove this it is only necessary to prove that for every positive root introduced into an equation there is one variation of sign added.

Suppose the signs of a quantic to be

$$+ \quad - \quad + \quad + \quad + \quad - \quad - \quad +,$$

and introduce a new positive root. We multiply by $x - h$, or, writing only the signs, by $+ \, -$. The result is

$$\begin{array}{ccccccccc}
+ & - & + & + & + & - & - & + & \\
+ & - & & & & & & & \\
\hline
+ & - & + & + & + & - & - & + & \\
 & - & + & - & - & - & + & + & - \\
\hline
+ & - & + & \pm & \pm & - & \mp & + & -
\end{array}$$

The ambiguous signs $\pm$, $\pm$ indicate that there is doubt whether the term is positive or negative. Examining the product, we see that to permanences in the multiplicand correspond ambiguities in the product. Hence, we cannot have a greater number of permanences in the product than in the multiplicand, and may have a less number. But there is one more term in the product than in the multiplicand, and this term always adds a new variation. Hence, we have *at least* one more *variation* in the product than in the multiplicand.

For each positive root introduced we have at least one more variation of sign. Hence, the number of positive roots cannot exceed the number of variations of sign.

Negative Roots. Change x to $-x$. The negative roots of the given equation are positive roots of this latter equation.

561. Hence, from Descartes' rule we obtain the following:

If the signs of the terms of an equation are all positive, the equation has no positive root.

If the signs of the terms of a complete equation are alternately positive and negative, the equation has no negative root.

If the roots of a complete equation are all real, the number of positive roots is the same as the number of variations of sign, and the number of negative roots is the same as the number of permanences of sign.

562. Existence of Complex Roots. In an incomplete equation Descartes' rule sometimes enables us to detect the presence of complex roots.

Thus, the equation $$x^3 + 5x + 7 = 0$$
may be written $$x^3 \pm 0x^2 + 5x + 7 = 0.$$

We are at liberty to assume that the second term is positive, or that it is negative.

Taking it positive, we have the signs

$$+ \quad + \quad + \quad +;$$

there is no variation, and the equation has no positive root.

Taking it negative, we have the signs

$$+ \quad - \quad + \quad +;$$

there is but one permanence and, therefore, not more than one negative root.

As there are three roots, and as complex roots enter in pairs, the given equation has one real negative root and two complex roots.

Exercise 86

All the roots of the equations given below are real; determine their signs.

1. $x^4 + 4x^3 - 43x^2 - 58x + 240 = 0.$
2. $x^3 - 22x^2 + 155x - 350 = 0.$
3. $x^4 + 4x^3 - 35x^2 - 78x + 360 = 0.$
4. $x^3 - 12x^2 - 43x - 30 = 0.$
5. $x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12 = 0.$
6. $x^3 - 12x^2 + 47x - 60 = 0.$
7. $x^4 - 2x^3 - 13x^2 + 38x - 24 = 0.$
8. $x^5 - x^4 - 187x^3 - 359x^2 + 186x + 360 = 0.$
9. $x^6 - 10x^5 + 19x^4 + 110x^3 - 536x^2 + 800x - 384 = 0.$
10. If an equation involves only even powers of x, and the signs are all positive, the equation has no real root, except 0.

11. If an equation involves only odd powers of x, and the signs are all positive, the equation has the root 0, and no other real root.

12. Show that the equation $x^6 - 3x^2 - x + 1 = 0$ has at least two complex roots.

13. Show that the equation $x^4 + 15x^2 + 7x - 11 = 0$ has two complex roots, and determine the signs of the real roots.

14. Show that the equation $x^3 + qx + r = 0$ has one negative root and two complex roots when q and r are both positive; and determine the character of the roots when q is negative and r positive.

15. Show that the equation $x^n - 1 = 0$ has but two real roots, $+1$ and -1, when n is even; and but one real root, $+1$, when n is odd.

16. Show that the equation $x^n + 1 = 0$ has no real root when n is even; and but one real root, -1, when n is odd.

563. Limits of the Roots. In solving numerical equations it is often desirable to obtain numbers between which the roots lie. Such numbers are called **limits of the roots.**

A *superior limit* to the positive roots of an equation is a number greater than any positive root. An *inferior limit* to the positive roots of an equation is a positive number less than any positive root.

General methods for finding limits to the roots are given in most text-books; but in practice close limits are more easily found as follows:

(1) $$x^4 - 5x^3 + 40x^2 - 8x + 23 = 0.$$

Writing this $\quad x^3(x-5) + 8x(5x-1) + 23 = 0,$

we see that the left member is positive for all values of x as great as 5; consequently, it cannot become 0 for any value as great as 5, and there is no root as great as 5.

(2) $$x^5 + 3x^4 + x^3 - 8x^2 - 51x + 18 = 0.$$

Writing this $x^2(x^3 - 8) + 3x(x^3 - 17) + x^3 + 18 = 0,$
we see that the left member is positive for all values of x as great as 3; consequently, there is no positive root as great as 3.

Sometimes we can find close limits by distributing the highest positive powers of x among the negative terms.

(3) $$x^4 + x^3 - 2x^2 - 4x - 24 = 0.$$

Multiply by 2, $2x^4 + 2x^3 - 4x^2 - 8x - 48 = 0.$

Writing this $x^2(x^2 - 4) + 2x(x^2 - 4) + x^4 - 48 = 0,$
we see that there is no positive root as great as 3.

An inferior limit to the positive roots is found by putting $x = \frac{1}{y}$ (§ 550), and then finding a superior limit to the positive roots of the transformed equation.

Limits to the *negative* roots of the equation $f(x) = 0$ are found by finding limits to the *positive* roots of the equation $f(-x) = 0$ (§ 547).

Exercise 87

Find superior limits to the positive roots of:

1. $x^3 - 2x^2 + 4x + 3 = 0.$
2. $2x^4 - x^2 - x + 1 = 0.$
3. $3x^4 + 5x^3 - 12x^2 + 10x - 18 = 0.$
4. $4x^4 - 3x^3 - x^2 + 7x + 5 = 0.$
5. $x^4 - x^3 - 2x^2 - 4x - 24 = 0.$
6. $4x^5 - 8x^4 + 22x^3 + 90x^2 - 60x + 1 = 0.$
7. $5x^5 + 14x^4 - 7x^3 + 12x^2 - 24x + 2 = 0.$
8. $2x^5 + 7x^4 + 5x^3 - 8x^2 - 4x + 3 = 0.$

CHAPTER XXXI

NUMERICAL EQUATIONS

564. A real root of a numerical equation is either **commensurable** or **incommensurable.**

Commensurable roots are either integers or fractions. Recurring decimals can be expressed as fractions (§ 280), and roots in that form are consequently commensurable.

Incommensurable roots cannot be found exactly, but may be calculated to any desired degree of accuracy by the method of approximation explained in this chapter.

COMMENSURABLE ROOTS

565. Integral Roots. The process of finding integral roots given in § 520 is long and tedious when there are many numbers to be tried. The number of divisors to be tried may be diminished by the following theorem:

Every integral root of an equation with integral coefficients is a divisor of the last term.

Let h be an integral root of the equation

$$ax^n + bx^{n-1} + cx^{n-2} + \cdots + dx^2 + ex + f = 0,$$

where the coefficients $a, b, c, \cdots, d, e$ are all integers.

Since h is a root,

$$ah^n + bh^{n-1} + ch^{n-2} + \cdots + dh^2 + eh + f = 0, \quad (\S\ 511)$$

or $\qquad f = - eh - dh^2 - \cdots - ch^{n-2} - bh^{n-1} - ah^n.$

Divide by h, $\dfrac{f}{h} = - e - dh - \cdots - ch^{n-3} - bh^{n-2} - ah^{n-1}.$

Since the right member is an integer, the left member must be an integer. That is, f is divisible by h.

Hence, in applying the method of § 520, we need try only divisors of the last term. The necessary labor may be still further reduced by the method of the following section.

566. Newton's Method of Divisors. In the above equation $\frac{f}{h}$ is an integer. Put $\frac{f}{h} = E$, transpose $-e$, and divide by h.

Then, $$\frac{E+e}{h} = -d - \cdots - ch^{n-4} - bh^{n-3} - ah^{n-2}.$$

Since the right member is an integer, $E + e$ must be divisible by h.

Put $\frac{E+e}{h} = D$, transpose $-d$, and divide by h.

Then, $$\frac{D+d}{h} = \cdots - ch^{n-5} - bh^{n-4} - ah^{n-3}.$$

As before, $D + d$ must be divisible by h.

By continuing the process we find that $C + c$, and $B + b$ are divisible by h, and for the last equation $\frac{B+b}{h} = -a$.

Transpose $-a$, $\frac{B+b}{h} + a = 0$, provided h is a root.

The preceding gives the following rule:

Divide the last term by h; *if the quotient is an integer, to it add the preceding coefficient, and again divide by* h; *if this quotient is an integer, add the preceding coefficient to it; and so on.*

If h is a root, the quotients are all integral, and the last sum is zero. A failure in either respect implies that h is not a root.

From the above we also obtain

$$E = -(ah^{n-1} + bh^{n-2} + ch^{n-3} + \cdots + dh + e),$$
$$D = -(ah^{n-2} + bh^{n-3} + ch^{n-4} + \cdots + d),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$C = -(ah^2 + bh + c),$$
$$B = -(ah + b),$$

so that the successive quotients, with their signs changed, are (§ 515), in reversed order, the coefficients of the quotient obtained by dividing the left member by $x - h$.

Find the integral roots of

$$3x^4 - 23x^3 + 42x^2 + 32x - 96 = 0.$$

By substitution we find that neither $+1$ nor -1 is a root.
The other divisors of -96 are ± 2, ± 3, ± 4, ± 6, etc.

Try $+2$.

$$\begin{array}{rrrrr|l} -96 & +32 & +42 & -23 & +3 & \underline{2} \\ & -48 & -8 & +17 & -3 & \\ \hline & -16 & +34 & -6 & 0 & \end{array}$$

Hence, $+2$ is a root. The coefficients of the depressed equation in reversed order are $-48 - 8 + 17 - 3$.

Try $+2$ again.

$$\begin{array}{rrrr|l} -48 & -8 & +17 & -3 & \underline{2} \\ & -24 & -16 & & \\ \hline & -32 & +1 & & \end{array}$$

Since 2 is not a divisor of $+1$, $+2$ is not again a root.

Try -2.

$$\begin{array}{rrrr|l} -48 & -8 & +17 & -3 & \underline{-2} \\ & +24 & -8 & & \\ \hline & +16 & +9 & & \end{array}$$

Since -2 is not a divisor of $+9$, -2 is not a root.

Try $+3$.

$$\begin{array}{rrrr|l} -48 & -8 & +17 & -3 & \underline{3} \\ & -16 & -8 & +3 & \\ \hline & -24 & +9 & 0 & \end{array}$$

Hence, $+3$ is a root. The depressed equation is

$$3x^2 - 8x - 16 = 0,$$

of which the roots are 4 and $-\frac{4}{3}$.

Therefore, the roots of the given equation are 2, 3, 4, $-\frac{4}{3}$.

The advantage of this method over that of § 520 is that if the number tried is not a root, this fact is detected as soon as we come to a fractional quotient; whereas, in § 520, we have to *complete* the division before we can decide whether or not the number tried is a root.

567. Fractional Roots. *A rational fraction cannot be a root of an equation with integral coefficients in the* p *form.*

If possible let $\frac{h}{k}$, where h and k are integers and $\frac{h}{k}$ is in its lowest terms, be a root.

Then, $$\frac{h^n}{k^n} + p_1\frac{h^{n-1}}{k^{n-1}} + p_2\frac{h^{n-2}}{k^{n-2}} + \cdots + p_n = 0.$$

Multiply by k^{n-1} and transpose,

$$\frac{h^n}{k} = -p_1h^{n-1} - p_2h^{n-2}k - \cdots - p_nk^{n-1}.$$

Now the right member is an integer; the left member is a fraction in its lowest terms, since h^n and k have no common divisor as h and k have no common divisor (§ 470, V). But a fraction in its lowest terms cannot be equal to an integer. Hence, $\frac{h}{k}$, or any other rational fraction, cannot be a root.

The real roots of an equation with integral coefficients in the p form are, therefore, integral or incommensurable.

If an equation has fractional roots, we can find these roots as follows:

Transform the equation into an equation with integral coefficients by multiplying the roots by some number m (§ 548). *Find the integral roots of the transformed equation and divide each by* m.

Solve the equation $36x^4 - 55x^2 - 35x - 6 = 0$.

Write this $$x^4 + 0x^3 - \tfrac{55}{36}x^2 - \tfrac{35}{36}x - \tfrac{1}{6} = 0.$$

Multiply the roots by 6,

$$x^4 - 55x^2 - 210x - 216 = 0,$$

of which the roots are found to be -2, -3, -4, $+9$.

Hence, the roots of the given equation are

$$-\tfrac{2}{6},\ -\tfrac{3}{6},\ -\tfrac{4}{6},\ +\tfrac{9}{6};\ \text{or},\ -\tfrac{1}{3},\ -\tfrac{1}{2},\ -\tfrac{2}{3},\ +\tfrac{3}{2}.$$

Exercise 88

Find the commensurable roots, and if possible all the roots, of each of the following equations:

1. $x^4 - 4x^3 - 8x + 32 = 0.$
2. $x^3 - 6x^2 + 10x - 8 = 0.$
3. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0.$
4. $x^3 + 3x^2 - 30x + 36 = 0.$
5. $x^4 - 12x^3 + 32x^2 + 27x - 18 = 0.$
6. $x^4 - 9x^3 + 17x^2 + 27x - 60 = 0.$
7. $x^5 - 5x^4 + 3x^3 + 17x^2 - 28x + 12 = 0.$
8. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$
9. $x^5 - 8x^4 + 11x^3 + 29x^2 - 36x - 45 = 0.$
10. $x^5 - x^4 - 6x^3 + 9x^2 + x - 4 = 0.$
11. $2x^4 - 3x^3 - 20x^2 + 27x + 18 = 0.$
12. $2x^4 - 9x^3 - 27x^2 + 134x - 120 = 0.$
13. $x^6 + 3x^5 - 2x^4 - 15x^3 - 15x^2 + 8x + 20 = 0.$
14. $18x^3 + 3x^2 - 7x - 2 = 0.$
15. $24x^3 - 34x^2 - 5x + 3 = 0.$
16. $27x^3 - 18x^2 - 3x + 2 = 0.$

INCOMMENSURABLE ROOTS

568. Location of the Roots. In order to calculate the value of an incommensurable root we must first find a rough approximation to the value of the root; for example, two integers between which it lies. This can generally be accomplished by successive applications of the principle of § 558. In some equations the methods of §§ 560–563 may be useful.

(1) Consider the equation $x^3 - 6x^2 + 3x + 5 = 0$.

We find (§ 510) $f(0) = +5$; $f(4) = -15$;
$f(1) = +3$; $f(5) = -5$;
$f(2) = -5$; $f(6) = +23$;
$f(3) = -13$; $f(-1) = -5$.

All numbers above 6 give $+$; all below -1 give $-$.

Hence (§ 558), the three roots are all real; one between 1 and 2; one between 5 and 6; one between 0 and -1.

(2) Find the first significant figure of each root of

$$x^4 - 2x^3 - 11x^2 + 6x + 2 = 0.$$

The equation has, by Descartes' rule (§ 560), not more than two positive roots and not more than two negative roots.

By (§ 510), $f(0) = +2$; $f(3) = -52$; $f(-1) = -12$;
$f(1) = -4$; $f(4) = -22$; $f(-2) = -22$;
$f(2) = -30$; $f(5) = +132$; $f(-3) = +20$.

Hence, there are two positive roots, one between 0 and 1, and one between 4 and 5; and two negative roots, one between 0 and -1, and one between -2 and -3. Plot the graph and get approximate values of the roots by measuring on the axis of x.

To find more closely a value for the root between 0 and 1, we find $f(0.5) = +2.06+$. Since $f(1) = -4$, the root lies between 0.5 and 1. We find $f(0.8) = -0.9+$. Hence, the root lies between 0.5 and 0.8. We find $f(0.7) = +0.4-$. Hence, the root lies between 0.7 and 0.8.

In a similar manner, we find the root between 0 and -1 to lie between -0.2 and -0.3.

Hence, the first significant figures of the roots are 0.7, 4, -0.2, -2.

Exercise 89

Determine the first significant figure of each real root of the following equations:

1. $x^3 - x^2 - 2x + 1 = 0$.
2. $x^3 - 5x - 3 = 0$.
3. $x^3 - 5x^2 + 7 = 0$.
4. $x^3 + 2x^2 - 30x + 39 = 0$.
5. $x^3 - 6x^2 - 3x + 5 = 0$.
6. $x^3 + 9x^2 + 24x + 17 = 0$.
7. $x^3 - 15x^2 + 63x - 50 = 0$.
8. $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0$.

569. Horner's Method of Approximation. By this method an incommensurable root may be found to any desired degree of approximation. We proceed to explain the method by applying it to one of the roots of the equation

$$x^3 - 6x^2 + 3x + 5 = 0. \qquad [1]$$

From Descartes' rule (§ 561) the equation has not more than two positive roots and not more than one negative root.

Before giving Horner's process we shall construct the graph of the function of x. In this way we not only locate the roots, but obtain a graphical representation which enables us to follow with ease the successive steps of the approximation, and to see exactly how they are made.

We will first compute a number of values of $f(x)$, writing these values in bold type.

VALUE OF x	VALUE OF $f(x)$	VALUE OF x	VALUE OF $f(x)$
0	+ **5**	+ 6	1 − 6 + 3 + 5 + 6 + 0 + 18 0 + 3 + **23**
+ 1	1 − 6 + 3 + 5 + 1 − 5 − 2 − 5 − 2 + **3**	− 1	1 − 6 + 3 + 5 − 1 + 7 − 10 − 7 + 10 − **5**
+ 2	1 − 6 + 3 + 5 + 2 − 8 − 10 − 4 − 5 − **5**	− 2	1 − 6 + 3 + 5 − 2 + 16 − 38 − 8 + 19 − **33**
+ 3	1 − 6 + 3 + 5 + 3 − 9 − 18 − 3 − 6 − **13**	− 3	1 − 6 + 3 + 5 − 3 + 27 − 90 − 9 + 30 − **85**
+ 4	1 − 6 + 3 + 5 + 4 − 8 − 20 − 2 − 5 − **15**	− 4	1 − 6 + 3 + 5 − 4 + 40 − 172 − 10 + 43 − **167**
	1 − 6 + 3 + 5 + 5 − 5 − 10 − 1 − 2 − **5**	− 5	1 − 6 + 3 + 5 − 5 + 55 − 290 − 11 + 58 − **285**

The points of maxima and minima are found by § 542.

$$f(x) = x^3 - 6x^2 + 3x + 5,$$

$$f'(x) = 3x^2 - 12x + 3 = 0;$$

whence, $$x = +3.73 \text{ or } +0.27,$$

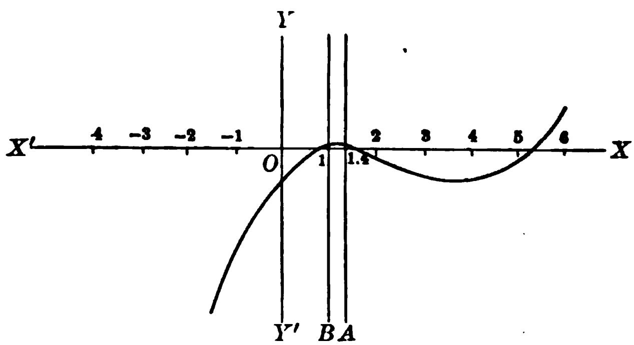
$$f''(x) = 6x - 12,$$

and is positive if $x = +3.73$, negative if $x = +0.27$.

For $x = 3.73$, $f(x) = -15.39$, a minimum.

For $x = 0.27$, $f(x) = +5.39$, a maximum.

The graph is plotted in the figure below, for convenience 5 spaces of coördinate paper being used for 1 horizontal unit, and 1 space for 5 vertical units.



We will now proceed to compute the positive root between 1 and 2 by Horner's Method. The graph shows that this root lies between 1 and 1.5.

Diminish the roots of the equation by 1; in other words, change the origin from its present position to the point marked 1 (§ 552). The numerical work is shown in the margin.

$$\begin{array}{rrrr|l}
1 & -6 & +3 & +5 & \underline{1} \\
 & +1 & -5 & -2 & \\
\hline
 & -5 & -2 & +3 & \\
 & +1 & -4 & & \\
\hline
 & -4 & -6 & & \\
 & +1 & & & \\
\hline
 & -3 & & &
\end{array}$$

The transformed equation is

$$x^3 - 3x^2 - 6x + 3 = 0, \qquad [2]$$

and its roots are less by 1 than those of the original equation. This fact is clearly shown by the new position of the origin.

As equation [1] has a root between 1 and 2, equation [2] must have a root between 0 and 1, the new zero of course being at the point marked 1 in the figure. The graph indicates that this root probably lies between 0 and 0.5 and nearer 0.5 than 0. The quickest way, therefore, to find the first figure of this root is to compute the value of $f(x)$ in [2] for different values of x, beginning with 0.5 and going backward 0.1 at a time till a change of sign occurs (§ 558). The numerical work is as follows:

$$\begin{array}{r|rrrr}
\underline{x = 0.5} & 1 & -3 & -6 & +3 \\
 & & +0.5 & -1.25 & -3.625 \\ \cline{2-5}
 & & -2.5 & -7.25 & -0.625 \\
\underline{x = 0.4} & 1 & -3 & -6 & +3 \\
 & & +0.4 & -1.04 & -2.816 \\ \cline{2-5}
 & & -2.6 & -7.04 & +0.184
\end{array}$$

Therefore, the second figure of the root we are seeking is 0.4.

We now diminish the roots of [2] by 0.4; that is, change the origin by an amount equal to 0.4 still farther towards the right. The new axis of y passes through the point marked 1.4.

The numerical work is as follows:

$$\begin{array}{rrrr|l}
1 & -3 & -6 & +3 & \underline{0.4} \\
 & +0.4 & -1.04 & -2.816 & \\ \cline{2-4}
 & -2.6 & -7.04 & +0.184 & \\
 & +0.4 & -0.88 & & \\ \cline{2-3}
 & -2.2 & -7.92 & & \\
 & +0.4 & & & \\ \cline{2-2}
 & -1.8 & & &
\end{array}$$

The second transformed equation is

$$x^3 - 1.8x^2 - 7.92x + 0.184 = 0. \qquad [3]$$

The roots of [3] are less by 0.4 than those of [2]. Since [2] has a root between 0.4 and 0.5, [3] must have a root between 0 and 0.1. As this root is much less than 1, the values of the terms in [3] containing powers of x higher than the first power must be very small; so that we shall probably obtain the first figure of the root, if we neglect the terms in [3] containing x^3 and x^2, and put

$$-7.92\,x + 0.184 = 0; \text{ whence, } x = 0.02 +.$$

Hence, the root of [1], which we are seeking, correct to two decimal places, is 1.4 + 0.02 or 1.42.

Diminish the roots of [3] by 0.02.

$$\begin{array}{rrrr|l}
1 & -1.8 & -7.92 & +0.184 & \underline{0.02} \\
 & +0.02 & -0.0356 & -0.159112 & \\
\hline
 & -1.78 & -7.9556 & +0.024888 & \\
 & +0.02 & -.00352 & & \\
\hline
 & -1.76 & -7.9908 & & \\
 & +0.02 & & & \\
\hline
 & -1.74 & & &
\end{array}$$

The third transformed equation is

$$x^3 - 1.74\,x^2 - 7.9908\,x + 0.024888 = 0. \qquad [4]$$

The roots of [4] are less by 0.02 than those of [3]. Since [3] has a root between 0.02 and 0.03, [4] must have a root between 0 and 0.01. This root is so much less than 1 that the first two, and even the first three, significant figures of it may be found by neglecting the powers of x higher than the first power and simply dividing the constant term by the coefficient of the first power of x.

$$-7.9908\,x + 0.024888 = 0.$$

$$x = \frac{0.024888}{7.9908} = 0.00311.$$

Therefore, the root of equation [1], correct to six significant figures, is

$$x = 1.42311.$$

This process may evidently be continued until the root is calculated to any desired degree of accuracy.

570. Remarks on Horner's Method.

First: We diminish the roots by a number *less* than the required root, and as we do not pass through the root, the sign of the last term remains unchanged throughout the work. The last coefficient but one always has a sign opposite to that of the last term.

If, in [3], the signs of the last two terms were alike, the value of x would be $-0.02+$. This would show that the value assumed for x was too great, and we should diminish the value of x and make the last transformation again.

The *first* transformation may, however, change the sign of the last term. Thus, if there had been a root between 0 and 1 in equation [1], diminishing the roots by 1 would have changed the sign of the last term.

Second: In finding the second figure of the root we make use of the theorem or change of sign (§ 558).

Any figure after the second is generally found correctly from the last two terms; for, in this case, the root is so small that powers of the root higher than the first are so much smaller than the root itself that the terms in which they appear have but slight influence upon the result.

571. It is not necessary to write the successive transformed equations. When the coefficients of any transformed equation have been computed, the next figure of the root may be found by dividing the last coefficient by the preceding coefficient, and changing the sign of the quotient.

Thus, in equation [4], the next figure of the root is obtained by dividing 0.024888 by 7.9908.

For this reason, the last coefficient but one of each transformed equation is called a **trial divisor**.

Sometimes the last coefficient but one in one of the transformed equations is zero. To find the next figure of the root in this case follow the method given for finding the second figure of the root.

The work may now be collected and arranged as follows:

1	− 6	+ 3	+ 5	1.423
	+ 1	− 5	− 2	
	− 5	− 2	**+ 3**	
	+ 1	− 4		
	− 4	**− 6**		
	+ 1			
1	**− 3**	− 6	+ 3	0.4
	+ 0.4	− 1.04	− 2.816	
	− 2.6	− 7.04	**+ 0.184**	
	+ 0.4	− 0.88		
	− 2.2	**− 7.92**		
	+ 0.4			
1	**− 1.8**	− 7.92	+ 0.184	0.02
	+ 0.02	− 0.0356	− 0.159112	
	− 1.78	− 7.9556	**+ 0.024888**	
	+ 0.02	− 0.0352		
	− 1.76	**− 7.9908**		
	+ 0.02			
1	**− 1.74**	− 7.9908	+ 0.024888	0.003

The numbers in heavy type are the coefficients of the successive transformed equations, the first coefficient of each equation being the same as the first coefficient of the given equation. In this example the first coefficient is 1.

When we have obtained the root to three places of decimals we can generally obtain two or three more figures of the root by simple division.

572. In practice it is convenient to avoid the use of the decimal points. We can do this as follows: multiply the roots of the first transformed equation by 10, the roots of the second transformed equation by 100, and so on. In the last example the first transformed equation now is

$$x^3 - 30x^2 - 600x + 3000 = 0,$$

and this equation has a root between 4 and 5. The second transformed equation now is

$$x^3 - 180x^2 - 79{,}200x + 184{,}000 = 0,$$

and this equation has a root between 2 and 3. And so on.

The complete work of the last example, for six figures of the root, is as follows:

1	− 6	+ 3	+ 5	1.42311 +
	+ 1	− 5	− 2	
	− 5	− 2	+ 3	
	+ 1	− 4		
	− 4	− 6		
	+ 1			
1	− 30	− 600	+ 3000	4
	+ 4	− 104	− 2816	
	− 26	− 704	+ 184	
	+ 4	− 88		
	− 22	− 792		
	+ 4			
1	− 180	− 79200	+ 184000	2
	+ 2	− 356	− 159112	
	− 178	− 79556	+ 24888	
	+ 2	− 352		
	− 176	− 79908		
	+ 2			
1	− 1740	− 7990800	+ 24888000	3
	+ 3	− 5211	− 23988033	
	− 1737	− 7996011	+ 899967	
	+ 3	− 5202		
	− 1734	− 8001213		
	+ 3			
1	− 17310	− 800121300	+ 899967000	1
	+ 1	− 17309	− 800138609	
	− 17309	− 800138609	+ 99828391	
	+ 1	− 17308		
	− 17308	− 800155917		
	+ 1			
1	− 173070	− 80015591700	+ 99828391000	1

We have here performed the work in full for six figures of the root. We can find five more figures of the root by simple division. If we divide 99,828,391 by 800,155,917, we obtain 0.124761, so that the required root to ten places of decimals is 1.4231124761.

The reason why simple division gives five more figures of the root is seen by examining the last transformed equation. Write this

$$8.00155917\,x = 0.000099828391 - 1.7307\,x^2 + x^3.$$

As x is about 0.00001, x^2 is about 0.0000000001, and x^3 is much smaller. Hence, the error in neglecting the x^2 and x^3 terms is in $8x$ about 0.00000000017, and in x about 0.00000000002. The result obtained by division is therefore correct to ten places of decimals.

Comparing the work on page 479 with that on page 478, we see that we have avoided the use of the decimal point by adopting the following rule:

When the coefficients of a transformed equation have been obtained, add one cipher to the second coefficient, two ciphers to the third coefficient, and so on. The coefficients and the next figure of the root are then integers.

If the root of the given equation lay between 0 and 1, we should *begin* by multiplying the roots of the equation by 10.

573. Negative Roots. To avoid the inconvenience of working with negative numbers, when we wish to calculate a negative root we change the signs of the roots (§ 547) and calculate the corresponding positive roots of the transformed equation.

Thus, one root of the equation

$$x^3 - 6x^2 + 3x + 5 = 0$$

lies between 0 and -1 (§ 568). By Horner's Method we find the corresponding root of

$$x^3 + 6x^2 + 3x - 5 = 0$$

to be 0.6696+. Hence, the required root of the given equation is $-0.6696+$.

Exercise 90

Compute to three decimal places for each of the following equations the root of which the first figure is the number in parenthesis opposite the equation:

1. $x^3 + 3x - 5 = 0.$ (1)
2. $x^3 - 6x - 12 = 0.$ (3)

3. $x^3 + x^2 + x - 100 = 0.$ (4)

4. $x^3 + 10x^2 + 6x - 120 = 0.$ (2)

5. $x^3 + 9x^2 + 24x + 17 = 0.$ (−4)

6. $x^4 - 12x^2 + 12x - 3 = 0.$ (−1)

7. $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0.$ (−0)

574. Contraction of Horner's Method. In § 572 the reader will see that if we seek only the first six figures of the root, the last six figures of the fourth coefficient of the last transformed equation may be rejected without affecting the result. Those figures of the second and third coefficients which enter into the fourth coefficient only in the rejected figures may also be rejected. Moreover, we may reject all the figures which stand in vertical lines over the figures already rejected.

The work may now be arranged as follows:

1	−6	+3	+5	1.42311+
	+1	−5	−2	
	−5	−2	+3000	
	+1	−4	−2816	
	−4	−600	+184000	
	+1	−104	−159112	
	−30	−704	+24888	
	+4	−88	−23991	
	−26	−79200	+897	
	+4	−356	−800	
	−22	−79556	+97	
	+4	−352	−80	
	−180	−79908		
	+2	−7991		
	−178	−6		
	+2	−7997		
	−176	−6		
	+2	−8003		
	−174	−800		
	−2	−80		

The double lines in the first column indicate that beyond this stage of the work the first column disappears altogether.

In the present example we first find three figures of the root. We then contract the work as follows:

Instead of adding ciphers to the coefficients of the transformed equation, we leave the last term as it is; from the last coefficient but one we strike off the last figure; from the last coefficient but two we strike off the last two figures; and so on. In each case we take for the remainder the nearest integer.

Thus, in the first column of the preceding example we strike off from 174 the last two figures, and take for the remainder 2 instead of 1.

The contracted process soon reduces to simple division.

Thus, in the last example, the last two figures of the root were found by simply dividing 897 by 800.

To insure accuracy in the last figure, the last divisor must consist of at least two figures. Consider the trial divisor at any stage of the work. If we begin to contract, we strike off one figure from the trial divisor *before* finding the next figure of the root. Since the last divisor is to consist of two figures, the contracted process will give us two less figures than there are figures in the trial divisor.

Thus, in § 572, if we begin to contract at the third trial divisor, $-79,908$, we can obtain three more figures of the root; if we begin to contract at the fourth trial divisor, $-8,001,213$, we can obtain five more figures of the root; and so on.

575. When the root sought is a large number we cannot find the successive figures of its *integral* portion by dividing the absolute term by the preceding coefficient, because the neglect of the higher powers, which are in this case large numbers, leads to serious error.

Find one root of $x^4 - 3x^2 + 11x - 4{,}842{,}624{,}131 = 0$.

$$x^4 - 3x^2 + 11x - 4{,}842{,}624{,}131 = 0. \qquad [1]$$

By trial, we find that a root lies between 200 and 300.

Diminish the roots of [1] by 200,

$$x^4 + 800x^3 + 239{,}997x^2 + 31{,}998{,}811x - 3{,}242{,}741{,}931 = 0. \qquad [2]$$

$$\text{If } x = 60, \quad f(x) = -273{,}064{,}071.$$
$$\text{If } x = 70, \quad f(x) = +471{,}570{,}139.$$

The signs of $f(x)$ show that a root lies between 60 and 70.

Diminish the roots of [2] by 60,

$$x^4 + 1040x^3 + 405{,}597x^2 + 70{,}302{,}451x - 273{,}064{,}071 = 0. \qquad [3]$$

The root of this equation is found by trial to lie between 3 and 4. Diminishing the roots by 3, we may find the remaining figures of the root by the usual process.

576. Any root of a positive number can be extracted by Horner's Method.

(1) Find the fourth root of 1296.

Here, $x^4 = 1296$,

or $x^4 + 0x^3 + 0x^2 + 0x - 1296 = 0$.

Calculate the root, $x = 6$.

If the number is a perfect power, the root is obtained exactly.

(2) Find the fourth root of 473.

Here, $x^4 = 473$,

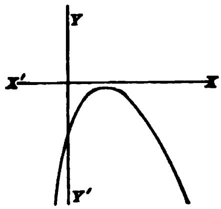
or $x^4 + 0x^3 + 0x^2 + 0x - 473 = 0$.

Calculate the root, $x = 4.66353+$.

577. Roots nearly Equal. In the preceding examples the changes of sign in the value of $f(x)$ enable us to determine the situation of the roots. In rare cases two roots may be so nearly equal that they both lie between the same two consecutive integers. In this case the existence of the roots will not be indicated by a change of sign in $f(x)$, and we must resort to other means to detect their presence.

Find the roots of the equation $x^3 - 515x^2 + 1155x - 649 = 0$.

$$x^3 - 515x^2 + 1155x - 649 = 0. \qquad [1]$$



By Descartes' rule this equation has no negative root. It has, therefore, certainly one, and perhaps three, positive roots.

We find $f(-1) = -2320$;
$f(0) = -649$;
$f(1) = -8$;
$f(2) = -391$;
$f(3) = -1792$.

The approach of $f(x)$ towards 0 indicates either that there are two roots near 1 or that the function approaches 0 without reaching it, the graph in the latter case being as here shown.

Let us proceed on the supposition that two roots near 1 do exist. Diminish the roots by 1. The transformed equation

$$x^3 - 512x^2 + 128x - 8 = 0, \qquad [2]$$

by Descartes' rule, still has either one or three positive roots, so that we have not passed the roots.

If we diminish the roots by 2, we obtain

$$x^3 - 509x^2 - 893x - 391 = 0,$$

which has but one positive root; so that we have passed both roots.

To find the second figure of the root, neglect the first term of equation [2]. Since the roots are nearly equal, the expression

$$512x^2 - 128x + 8$$

must be nearly a perfect square. Comparing this with $a(x - \alpha)^2$, or $ax^2 - 2a\alpha x + a\alpha^2$, we see that $\frac{128}{2 \times 512}$ and $\frac{2 \times 8}{128}$ are approximate values for the roots; these both give $\frac{1}{8}$, or 0.12.

Diminish the roots by 0.1; the work is as before. Continue until the two quotients obtained as above give different numbers for the next figure of the root. In the present example this occurs when we come to the third decimal figure; the transformed equation is

$$x^3 - 51{,}164x^2 + 51{,}632x - 11{,}072 = 0, \qquad [3]$$

and the two quotients are 0.5+ and 0.3+. To separate the roots, try 0.4; the left member of the last equation is found to be +. Since 0 gives − and 1 gives −, there is one root between 0 and 0.4, and one between 0.4 and 1.

To calculate the first root, we try 0.3; as this gives a − sign, we diminish the roots by 0.3 and proceed as in § 574.

1	− 515	+ 1155	− 649	1.1230914
	1	− 514	+ 641	
	− 514	+ 641	− 8000	
	1	− 513	+ 7681	
	− 513	+ 12800	− 319000	
	1	− 5119	+ 307928	
	− 5120	+ 7681	− 11072000	
	1	− 5118	+ 10884867	
	− 5119	+ 256300	− 187133	
	1	− 102336	+ 184275	
	− 5118	+ 153964	− 2858	
	1	− 102332	+ 2002	
	− 51170	+ 5163200	− 856	
	2	− 1534911	+ 800	
	− 51168	+ 3628289		
	2	− 1534902		
	− 51166	+ 2093387		
	2	+ 209339		
	− 511640	+ 20934		
	3	− 459		
	− 511637	+ 20475		
	3	− 459		
	− 511634	+ 20016		
	3	+ 2002		
	− 511631	+ 200		
	− 5116			
	− 51			

To calculate the second root, we return to equation [3],

$$x^3 - 51,164\,x^2 + 51,632\,x - 11,072 = 0.$$

We have $f(0.4) = +$, $f(1) = -$; we find $f(0.6) = +$, $f(0.7) = +\,0.383$. Since $f(0.7)$ is so small, $f(0.8)$ is undoubtedly negative.

Diminish the roots by 7 and proceed as follows:

1	− 511640	+ 5163200	− 11072000	1.1270002
	7	− 3581431	+ 11072383	
	− 511633	+ 1581769	+ 383	
	7	− 3581382		
	− 511626	− 1999613		
	7	− 200		
	− 511619			

Since the sum of the roots (§ 521) is 515, we can find the third root by subtracting from 515 the sum of the two roots already found.

Hence, the 3d root $= 515 - (1.1230914 + 1.1270002) = 512.7499084$.

578. From the preceding sections we obtain the following general directions for solving a numerical equation:

1. Find and remove commensurable roots by §§ 565–567, if there are any such roots in the equation.

2. Determine the situation and thence the first figure of each of the incommensurable roots as in § 568.

3. Calculate the incommensurable roots by Horner's Method.

Exercise 91

Calculate to six places of decimals the positive roots of the following equations:

1. $x^3 - 3x - 1 = 0.$
2. $x^3 + 2x^2 - 4x - 43 = 0.$
3. $3x^3 + 3x^2 + 8x - 32 = 0.$
4. $2x^3 - 26x^2 + 131x - 202 = 0.$
5. $x^4 - 12x + 7 = 0.$
6. $x^4 - 5x^3 + 2x^2 - 13x + 55 = 0.$

Calculate, to six places of decimals where incommensurable, the real roots of the following equations:

7. $x^3 = 35{,}499.$
8. $x^3 = 242{,}970{,}624.$
9. $x^4 = 707{,}281.$
10. $x^5 = 147{,}008{,}443.$
11. $x^3 + 2x + 20 = 0.$
12. $x^3 - 10x^2 + 8x + 120 = 0.$

Each of the following equations has two roots nearly equal. Calculate the roots to six places of decimals:

13. $x^3 - 3x^2 - 4x + 13 = 0.$
14. $2x^4 + 8x^3 - 35x^2 - 36x + 117 = 0.$
15. $x^3 + 11x^2 - 102x + 181 = 0.$

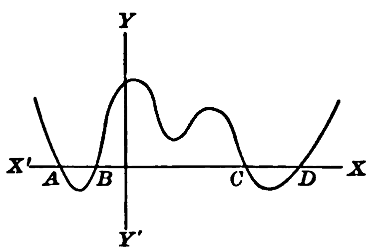
STURM'S THEOREM

579. The problem of finding the number and the situation of the real roots of an equation is completely solved by Sturm's Theorem. In theory Sturm's method is perfect; in practice its application is long and tedious. For this reason, the situation of the roots is in general more easily determined by the methods already given.

Before passing on to Sturm's Theorem itself, we shall prove two preliminary theorems.

580. Situation of the Roots of $f'(x) = 0$. *Between any two distinct real roots of the equation* $f(x) = 0$ *there lies at least one real root of the equation* $f'(x) = 0$.

Let α and β be two real roots of $f(x) = 0$, β being greater than α. Then $f(\alpha) = 0$ and $f(\beta) = 0$. As x increases continuously from α to β, $f(x)$ changes from 0 to 0 again; and must first increase and then decrease, or first decrease and then increase. Hence, there must be some point at which $f'(x)$ changes from $+$ to $-$, or *vice versa*. Therefore, for some value of x between α and β, $f'(x)$ must be zero. Hence, at least one root of $f'(x) = 0$ must lie between α and β.



In the graph the curve will be horizontal where $f'(x) = 0$. In the figure here given, A, B, C, D correspond to roots of $f(x) = 0$. Between A and B there is one root of $f'(x) = 0$; between B and C, three roots; and between C and D, one root.

It is evident that if more than one root of $f'(x)$ lies between α and β, the number of roots must be an *odd* number.

581. Signs of f(x) and f'(x). *Let α be any real root of an equation,* $f(x) = 0$, *which has no equal roots.*

Let x *change continuously from* $\alpha - h$, *a value a little less than* α, *to* $\alpha + h$, *a value a little greater than* α. *Then* $f(x)$ *and* $f'(x)$ *have unlike signs immediately before* x *passes through the root, and like signs immediately after* x *passes through the root.*

For, $$f(\alpha - h) = - hf'(\alpha) + \frac{h^2}{\underline{|2}} f''(\alpha) - \cdots,$$

and $$f'(\alpha - h) = f'(\alpha) - hf''(\alpha) + \cdots; \qquad (\S\ 544)$$

since $f(\alpha) = 0$, as α is a root of $f(x) = 0$.

When h is very small the sign of each series on the right is the sign of its first term (§ 556); and $f(\alpha - h)$ and $f'(\alpha - h)$ evidently have opposite signs.

Similarly, $f(\alpha + h)$ and $f'(\alpha + h)$ have like signs.

NOTE. The above is also evident from the graph of $f(x)$.

582. Sturm's Functions. The process of finding the H.C.F. of $f(x)$ and $f'(x)$ has been employed (§ 543) in obtaining the multiple roots of the equation $f(x) = 0$. We use the same process in Sturm's Method.

Let $f(x) = 0$ be an equation which has no multiple roots; let the operation of finding the H.C.F. of $f(x)$ and $f'(x)$ be carried on until the remainder does not involve x, *the sign of each remainder obtained being changed before it is used as a divisor.*

NOTE. If there is a H.C.F., the equation has multiple roots. Remove them and proceed with the reduced equation.

Represent by $f_2(x), f_3(x), \cdots, f_n(x)$ the several remainders with their signs changed. These expressions with $f'(x)$ are called **Sturm's Functions.**

Now, if D represents the dividend, d the divisor, q the quotient, and R the remainder,

$$D \equiv qd + R.$$

Consequently,
$$\begin{aligned} f(x) &\equiv q_1 f'(x) - f_2(x), \\ f'(x) &\equiv q_2 f_2(x) - f_3(x), \\ f_2(x) &\equiv q_3 f_3(x) - f_4(x), \\ &\cdots\cdots \\ &\cdots\cdots \\ f_{n-2}(x) &\equiv q_{n-1} f_{n-1}(x) - f_n(x), \end{aligned}$$

where $q_1, q_2, \cdots, q_{n-1}$ represent the several quotients, or the quotients multiplied by positive integers.

From the above identities we have the following:

1. Two consecutive functions cannot vanish for the same value of x.

For example, suppose that $f_2(x)$ and $f_3(x)$ vanish for a particular value of x. Give to x this value in all the identities. By the third identity, $f_4(x)$ will vanish; by the fourth, $f_5(x)$ will vanish; finally, $f_n(x)$ will vanish, which is contrary to the hypothesis that $f(x) = 0$ has no multiple roots.

2. When we give to x a value which causes any one function to vanish, the adjacent functions have opposite signs.

Thus, if $f_3(x) = 0$, from the third identity $f_2(x) = -f_4(x)$.

583. Sturm's Theorem. We are now in a position to enunciate Sturm's Theorem:

If in the series of functions

$$\mathrm{f}(\mathrm{x}),\quad \mathrm{f}'(\mathrm{x}),\quad \mathrm{f}_2(\mathrm{x}),\quad \cdots,\quad \mathrm{f}_n(\mathrm{x})$$

we give to x *any particular value* a, *and determine the number of variations of sign; then give to* x *any greater value* b, *and determine the number of variations of sign; the number of variations lost is the number of real roots of the equation* $\mathrm{f}(\mathrm{x}) = 0$ *between* a *and* b.

For, let x increase continuously from a to b.

1. Take the case in which x passes through a root of any of the functions $f'(x), f_2(x), \cdots, f_{n-1}(x)$, for example $f_4(x)$.

The adjacent functions have opposite signs. $f_4(x)$ itself changes sign, but this has no effect on the number of variations; for if just before x passes through the root the signs are $+ + -$, just after x passes through the root they are $+ - -$, and the number of variations is in each case one.

Hence, there is no change in the number of variations of sign when x passes through a root of any of the functions $f'(x), f_2(x), \cdots, f_{n-1}(x)$.

2. Take the case in which x passes through a root of $f(x) = 0$. Since $f(x)$ and $f'(x)$ have unlike signs just before x passes through the root, and like signs just after (§ 581), there is one variation lost for each root of $f(x) = 0$.

Hence, the number of real roots between a and b is the number of variations of sign lost as x passes from a to b.

To determine the number of real roots, we take x first very large and negative, and then very large and positive. The sign of each function is then the sign of its first term (§ 555).

The reader may not understand how it is that $f(x)$ and $f'(x)$ always have unlike signs just before x passes through a root.

Let α and β be two consecutive roots of $f(x) = 0$; let h be very small. Suppose that at α $f(x)$ changes from $+$ to $-$; then $f'(\alpha)$ is $-$ (§ 540).

When $\quad x = \alpha - h, \quad f(x)$ is $+, \quad f'(x)$ is $-$;

when $\quad x = \alpha, \quad f(x)$ is $0, \quad f'(x)$ is $-$.

As x changes from α to β, $f'(x)$ passes through an odd number of roots (§ 580), and consequently changes sign. Hence, when $x = \beta - h$, $f(x)$ is $-$, $f'(x)$ is $+$; and $f'(x)$ and $f(x)$ again have unlike signs.

584. Examples. (1) Determine the number and the signs of the real roots of the equation

$$x^4 - 4x^3 + 6x^2 - 12x + 1 = 0.$$

Here, $\quad f'(x) \equiv 4x^3 - 12x^2 + 12x - 12.$

Let us take for $f'(x)$, however, the simpler expression

$$x^3 - 3x^2 + 3x - 3.$$

We proceed as if to find the H.C.F., changing the sign of each remainder before using it as a divisor.

$1 - 3 + 3 - 3$	$1 - 4 + 6 - 12 + 1$	$1 - 1$
$3 - 9 + 9 - 9$	$1 - 3 + 3 - 3$	
$3 + 1$	$-1 + 3 - 9 + 1$	
$-10 + 9$	$-1 + 3 - 3 + 3$	
$-30 + 27$	$-6 - 2$	
$-30 - 10$	$3 + 1$	$1 - 10 + 37$
$37 - 9$		
$111 - 27$		
$111 + 37$		
-64		
$+64$		

The coefficients of the several functions are in heavy type. In the ordinary process of finding the H.C.F. we can change signs at pleasure. In finding Sturm's functions we cannot do this, as the sign is all important. We can, however, take out any positive factor.

We now have

$$f(x) \equiv x^4 - 4x^3 + 6x^2 - 12x + 1,$$
$$f'(x) \equiv x^3 - 3x^2 + 3x - 3,$$
$$f_2(x) \equiv 3x + 1,$$
$$f_3(x) \equiv + 64.$$

		$f(x)$	$f'(x)$	$f_2(x)$	$f_3(x)$	
When	$x = -1000$	$+$	$-$	$-$	$+$	2 variations.
	$x = 0$	$+$	$-$	$+$	$+$	2 variations.
	$x = +1000$	$+$	$+$	$+$	$+$	0 variations.

Hence, the equation has two real positive roots; it must therefore have two complex roots.

The real roots are found, by § 568, to lie one between 0 and 1, and one between 3 and 4.

(2) Investigate the character of the roots of the equation

$$x^3 + 3Hx + G = 0.$$

We find

$$f(x) \equiv x^3 + 3Hx + G,$$
$$f'(x) \equiv 3(x^2 + H),$$
$$f_2(x) \equiv -2Hx - G,$$
$$f_3(x) \equiv -(G^2 + 4H^3).$$

If $G^2 + 4H^3$ is *positive*, we have

	$f(x)$	$f'(x)$	$f_2(x)$	$f_3(x)$	
$x = -\infty$	$-$	$+$	$\pm$	$-$	2 variations.
$x = +\infty$	$+$	$+$	$\mp$	$-$	1 variation.

Since H may be either $+$ or $-$, the sign of $f_2(x)$ is ambiguous. Hence, when $G^2 + 4H^3$ is positive there is but one real root.

If $G^2 + 4H^3$ is *negative*, H must be negative, and we have

	$f(x)$	$f'(x)$	$f_2(x)$	$f_3(x)$	
$x = -\infty$	$-$	$+$	$-$	$+$	3 variations.
$x = +\infty$	$+$	$+$	$+$	$+$	0 variation.

Hence, when $G^2 + 4H^3$ is negative there are three real roots.

Exercise 92

Find by Sturm's Theorem the number and the situation of the real roots of the following equations:

1. $x^3 - 4x^2 - 11x + 43 = 0.$
2. $x^3 - 6x^2 + 7x - 3 = 0.$
3. $x^4 - 4x^3 + x^2 + 6x + 2 = 0.$
4. $x^4 - 5x^3 + 10x^2 - 6x - 21 = 0.$
5. $x^4 - x^3 - x^2 + 6 = 0.$
6. $x^4 - 2x^3 - 3x^2 + 10x - 4 = 0.$
7. $x^5 + 2x^4 + 3x^3 + 3x^2 - 1 = 0.$
8. $x^5 + x^3 - 2x^2 + 3x - 2 = 0.$
9. $x^4 - 12x^3 + 47x^2 - 66x + 27 = 0.$
10. $9x^4 - 54x^3 + 60x^2 - 72x + 16 = 0.$
11. $2x^4 - 5x^3 - 17x^2 + 53x - 28 = 0.$
12. $x^4 + 2x^3 - 37x^2 - 38x + 1 = 0.$
13. $121x^4 + 198x^3 - 100x^2 - 36x + 4 = 0.$

CHAPTER XXXII

GENERAL SOLUTION OF EQUATIONS

585. Numerical and Algebraic Solutions. By the methods of the preceding chapter we can find to any desired degree of accuracy the real roots of a numerical equation of any degree. The methods are theoretically complete, and the solution of a numerical equation becomes simply a question of the labor required for the necessary computations.

In the case of a literal equation we have an entirely different problem to solve. To solve a literal equation, we have to find in terms of the coefficients expressions which will, when substituted for the unknown in the given equation, reduce that equation to an identity. Thus, the roots of the general quadratic $ax^2 + bx + c = 0$ have been found to be (§ 191)

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In the case of a particular quadratic with numerical coefficients the roots can be found by putting for a, b, c in the above expression their particular values, and performing the indicated operations.

Similar solutions have been obtained for the general equations of the third and fourth degrees, and for certain special forms of equations of higher degrees.

The solution of the general equation of the fifth degree involves expressions called *elliptic functions*, and is consequently beyond the scope of the present treatise.

In many cases, however, the numerical values of the roots of a particular equation are not easily obtained from the

general solution, and for numerical equations the general solutions are in such cases of little value.

A general solution differs from the solutions obtained in the last chapter in that a general solution represents not one particular root but all the roots indiscriminately.

586. Reciprocal Equations. Reciprocal equations (§ 551), called also *recurring equations*, are of four forms:

1. Degree even; corresponding coefficients equal with like signs.

2. Degree even; corresponding coefficients numerically equal but with unlike signs.

3. Degree odd; corresponding coefficients equal with like signs.

4. Degree odd; corresponding coefficients numerically equal but with unlike signs.

The following are examples of the four forms:

1. $2x^4 - 3x^3 + 4x^2 - 3x + 2 = 0;$
2. $3x^6 - x^5 + 2x^4 - 2x^2 + x - 3 = 0;$
3. $x^5 + 3x^4 - 2x^3 - 2x^2 + 3x + 1 = 0;$
4. $2x^5 + 5x^4 + x^3 - x^2 - 5x - 2 = 0.$

Every equation of the second form evidently lacks the middle term.

Every reciprocal equation of the second, third, or fourth form can be depressed to an equation of the first form.

Second Form. Consider the equation

$$ax^6 + bx^5 + cx^4 - cx^2 - bx - a = 0,$$

or
$$a(x^6 - 1) + bx(x^4 - 1) + cx^2(x^2 - 1) = 0.$$

Hence, the equation is divisible by $x^2 - 1$; consequently, 1 and -1 are both roots. The depressed equation formed by dividing the given equation by $x^2 - 1$ is

$$ax^4 + bx^3 + (a + c)x^2 + bx + a = 0,$$

which is evidently of the first form.

Similarly for any equation of the second form.

Third Form. Consider the equation

$$ax^5 + bx^4 + cx^3 + cx^2 + bx + a = 0,$$

or $$a(x^5 + 1) + bx(x^3 + 1) + cx^2(x + 1) = 0. \qquad [1]$$

Hence, the equation is divisible by $x + 1$; consequently, -1 is a root. The depressed equation formed by dividing [1] by $x + 1$ is

$$ax^4 - (a - b)x^3 + (a - b + c)x^2 - (a - b)x + a = 0,$$

which is evidently of the first form.

Similarly for any equation of the third form.

Fourth Form. Consider the equation

$$ax^5 + bx^4 + cx^3 - cx^2 - bx - a = 0,$$

or $$a(x^5 - 1) + bx(x^3 - 1) + cx^2(x - 1) = 0. \qquad [1]$$

Hence, the equation is divisible by $x - 1$; consequently, $+1$ is a root. The depressed equation formed by dividing [1] by $x - 1$ is

$$ax^4 + (a + b)x^3 + (a + b + c)x^2 + (a + b)x + a = 0,$$

which is evidently of the first form.

Similarly for any equation of the fourth form.

By the preceding, to solve any reciprocal equation, it is only necessary to solve one of the first form.

587. *Any reciprocal equation of the first form can be depressed to an equation of half the degree.*

We proceed to illustrate by examples:

(1) Solve the equation $x^4 - 12x^3 + 29x^2 - 12x + 1 = 0$.

Divide by x^2, $$x^2 + \frac{1}{x^2} - 12\left(x + \frac{1}{x}\right) + 29 = 0.$$

Solve this equation for $x + \dfrac{1}{x}\cdot$

Then, $$x + \frac{1}{x} = 9 \text{ or } 3.$$

Solve these equations for x.

Then, $x = \frac{9 \pm \sqrt{77}}{2}$, and $x = \frac{3 \pm \sqrt{5}}{2}$.

The first two roots are reciprocals each of the other; also the second two roots are reciprocals each of the other.

(2) Solve the equation

$$x^5 - 3x^4 + 5x^3 - 5x^2 + 3x - 1 = 0.$$

This is of the fourth form; dividing by $x - 1$, we find the depressed equation to be

$$x^4 - 2x^3 + 3x^2 - 2x + 1 = 0.$$

This may be written

$$x^2 + 2 + \frac{1}{x^2} - 2\left(x + \frac{1}{x}\right) + 1 = 0,$$

or $$\left(x + \frac{1}{x}\right)^2 - 2\left(x + \frac{1}{x}\right) + 1 = 0.$$

Extract the root, $x + \frac{1}{x} - 1 = 0.$

Solve, $x = \frac{1 \pm \sqrt{-3}}{2}$,

these expressions being double roots.

Exercise 93

Solve the equations:

1. $x^4 + 7x^3 - 7x - 1 = 0.$
2. $x^4 + 2x^3 + x^2 + 2x + 1 = 0.$
3. $x^6 - 3x^5 + 5x^4 - 5x^2 + 3x - 1 = 0.$
4. $x^4 - 5x^3 + 6x^2 - 5x + 1 = 0.$
5. $2x^4 - 5x^3 + 6x^2 - 5x + 2 = 0.$
6. $x^5 - 4x^4 + x^3 + x^2 - 4x + 1 = 0.$
7. $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0.$
8. $x^3 + mx^2 + mx + 1 = 0.$
9. $x^5 + x^4 - x^3 - x^2 + x + 1 = 0.$
10. $3x^5 - 2x^4 + 5x^3 - 5x^2 + 2x - 3 = 0.$

588. Binomial Equations. An equation of the form

$$x^n \pm a = 0$$

is called a **binomial equation.**

We shall first consider the two equations

$$x^n - 1 = 0, \quad x^n + 1 = 0.$$

If n is *even*, the equation $x^n + 1 = 0$, by Descartes' rule (§ 560), has no real roots; the equation $x^n - 1 = 0$ has two real roots, $+1$ and -1, the remaining $n - 2$ roots being complex.

If n is *odd*, the equation $x^n + 1 = 0$ has one real root, -1; the equation $x^n - 1 = 0$ has one real root, $+1$, the remaining $n - 1$ roots being in each case complex.

589. Now consider the equation $x^n \pm a = 0$, where a is positive. Represent by $\sqrt[n]{a}$ the positive scalar nth root of a. Then, if α is any root of $x^n \pm 1 = 0$, $\alpha \sqrt[n]{a}$ will be a root of $x^n \pm a = 0$.

For, $$(\alpha \sqrt[n]{a})^n = \alpha^n a = \mp 1 \times a = \mp a.$$

Since α is any root of $x^n \pm 1 = 0$, the n roots of $x^n \pm a = 0$ are found by multiplying each of the n roots of $x^n \pm 1 = 0$ by $\sqrt[n]{a}$.

The roots of a binomial equation are all different. For $x^n \pm a$ and its derivative nx^{n-1} can have no common factor involving x (§ 543).

590. *If* α *is a root of the equation* $x^n - 1 = 0$, *then* α^k, *where* k *is an integer, is also a root.*

For, if α is a root, $\alpha^n = 1$.

But $$(\alpha^k)^n = (\alpha^n)^k = (1)^k = 1.$$

Therefore, α^k is a root of $x^n = 1$, or of $x^n - 1 = 0$.

Similarly for a root of $x^n + 1 = 0$, provided k is an odd integer.

591. The Cube Roots of Unity. The equation $x^3 = 1$, or $x^3 - 1 = 0$, may be written

$$(x-1)(x^2+x+1)=0,$$

of which the three roots are

$$1,\ -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-3},\ -\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-3}.$$

If either of the complex roots is represented by ω, the other is found by actual multiplication to be ω^2. This agrees with the last section.

Also, $$\omega^2+\omega+1=0.$$

In a similar manner, we find the roots of $x^3 = -1$ to be

$$-1,\ \tfrac{1}{2}-\tfrac{1}{2}\sqrt{-3},\ \tfrac{1}{2}+\tfrac{1}{2}\sqrt{-3},$$

or $$-1,\ -\omega,\ -\omega^2.$$

592. Examples. (1) Find the six sixth roots of 1.

We have to solve $$x^6-1=0,$$

or $$(x^3-1)(x^3+1)=0.$$

Hence, the roots are ± 1, $\pm\omega$, $\pm\omega^2$.

(2) Find the five fifth roots of 1.

We have to solve $$x^5-1=0,$$

or $$(x-1)(x^4+x^3+x^2+x+1)=0.$$

$$\therefore x-1=0, \text{ or } x=1;$$

or $$x^4+x^3+x^2+x+1=0,$$

or $$\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)+1=0.$$

Solve for $x+\dfrac{1}{x}$, $$x+\frac{1}{x}=\frac{-1\pm\sqrt{5}}{2}.$$

Solve these equations for x, and we obtain for the remaining four roots,

$$\frac{-1+\sqrt{5}\pm\sqrt{10+2\sqrt{5}}\sqrt{-1}}{4},\quad \frac{-1-\sqrt{5}\pm\sqrt{10-2\sqrt{5}}\sqrt{-1}}{4}.$$

Exercise 94

Solve the binomial equations:

1. $x^6 + 1 = 0$.
3. $x^9 - 1 = 0$.
2. $x^8 - 1 = 0$.
4. $x^5 - 243 = 0$.

5. Find the quintic on which depends the solution of the equation $x^{11} = 1$.

6. Show that $x^3 \pm y^3 \equiv (x \pm y)(x \pm \omega y)(x \pm \omega^2 y)$.

7. Show that

$$x^2 + y^2 + z^2 - yz - zx - xy \equiv (x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z).$$

8. If α is a complex root of $x^5 - 1 = 0$, show that

$$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5.$$

593. The General Cubic. We shall write the general equation of the third degree in the form

$$ax^3 + 3bx^2 + 3cx + d = 0. \qquad [1]$$

Before attempting to solve this equation we shall transform it into an equation in which the second term is wanting.

Put $z = ax + b$. Then, $x = \dfrac{z-b}{a}$.

Substitute this expression for x and reduce,

$$z^3 + 3(ac - b^2)z + (a^2d - 3abc + 2b^3) = 0,$$

or, putting $H \equiv ac - b^2$, and $G \equiv a^2d - 3abc + 2b^3$,

$$z^3 + 3Hz + G = 0. \qquad [2]$$

In the transformed equation put $z = u^{\frac{1}{3}} + v^{\frac{1}{3}}$.

Then, $\quad (u^{\frac{1}{3}} + v^{\frac{1}{3}})^3 + 3H(u^{\frac{1}{3}} + v^{\frac{1}{3}}) + G = 0,$

which reduces to

$$u + v + 3(u^{\frac{1}{3}}v^{\frac{1}{3}} + H)(u^{\frac{1}{3}} + v^{\frac{1}{3}}) + G = 0. \qquad [3]$$

Since we have assumed but one relation between u and v, we are at liberty to assume one more relation.

Let us assume $\qquad u^{\frac{1}{3}}v^{\frac{1}{3}} = -H.$ [4]

[3] now reduces to $\qquad u + v = -G.$ [5]

[4] may be written $\qquad uv = -H^3.$ [6]

Eliminate v from [5] and [6], $u^2 + Gu = H^3.$ [7]

Equation [7] is called the **reducing quadratic** of the cubic.

Solving this quadratic, we find

$$\left.\begin{aligned} u &= \frac{-G \pm \sqrt{G^2 + 4H^3}}{2} \\ v &= \frac{-H^3}{u} = \frac{-G \mp \sqrt{G^2 + 4H^3}}{2} \end{aligned}\right\}. \qquad [8]$$

Since $ax + b = z = u^{\frac{1}{3}} + v^{\frac{1}{3}}$, the three values of z are

$$u^{\frac{1}{3}} - \frac{H}{u^{\frac{1}{3}}}, \quad \omega u^{\frac{1}{3}} - \frac{H}{\omega u^{\frac{1}{3}}}, \quad \omega^2 u^{\frac{1}{3}} - \frac{H}{\omega^2 u^{\frac{1}{3}}},$$

where $u^{\frac{1}{3}}$ is any one of the three cube roots of u.

Since there is the sign $\pm$ before the radical, we have apparently six values of z. From [4] it is seen, however, that there are really but three different values of z.

The above solution is known as *Cardan's.*

Solve, by Cardan's Method,

$$2x^3 - 6x^2 + 12x - 11 = 0.$$

Here, $\qquad a = 2, \quad b = -2.$

Put $z = 2x - 2$; then $z^3 + 12z - 12 = 0.$

$\therefore H = 4,\ G = -12,$ and the reducing quadratic is

$$u^2 - 12u = 64.$$

Solve, $\qquad u = 6 \pm 10 = 16 \text{ or } -4.$

$$\therefore v = -\frac{H^3}{u} = -4 \text{ or } +16.$$

Hence, the values of z are

$$2\sqrt[3]{2}-\sqrt[3]{4};\ 2\omega\sqrt[3]{2}-\omega^2\sqrt[3]{4};\ 2\omega^2\sqrt[3]{2}-\omega\sqrt[3]{4};$$

and the values of x are

$$1+\sqrt[3]{2}-\frac{1}{\sqrt[3]{2}};\ 1+\omega\sqrt[3]{2}-\frac{\omega^2}{\sqrt[3]{2}};\ 1+\omega^2\sqrt[3]{2}-\frac{\omega}{\sqrt[3]{2}}.$$

594. Discussion of the Solution. The above solution, while complete as an algebraic solution, is of little value in solving numerical equations.

In the case of a cubic there are three cases to consider.

I. *All three roots real and unequal.* In this case G^2+4H^3 is negative (§ 584, Example 2), and its square root is orthotomic. If we put $K^2=-(G^2+4H^3)$, we have

$$ax+b=\left(\frac{-G+K\sqrt{-1}}{2}\right)^{\frac{1}{3}}+\left(\frac{-G-K\sqrt{-1}}{2}\right)^{\frac{1}{3}}.$$

Since there is no general algebraic rule for extracting the cube root of a complex expression, the case of three real and unequal roots is known as the *irreducible* case.

II. *Two of the roots equal.* In this case $G^2+4H^3=0$ (§ 584, Example 2), and we have

$$ax+b=\left(\frac{-G}{2}\right)^{\frac{1}{3}}+\left(\frac{-G}{2}\right)^{\frac{1}{3}}.$$

III. *Two roots complex.* In this case G^2+4H^3 is positive (§ 584, Example 2), its square root is real, and we have

$$ax+b=\left(\frac{-G+\sqrt{G^2+4H^3}}{2}\right)^{\frac{1}{3}}+\left(\frac{-G-\sqrt{G^2+4H^3}}{2}\right)^{\frac{1}{3}}.$$

The value of the expression G^2+4H^3 determines the nature of the roots. For this reason, G^2+4H^3 is called the **discriminant** of the cubic.

Hence, we conclude that the general solution gives the roots of a numerical cubic in a form in which their values can be readily computed only in the second and third cases.

In either of these cases, however, the real roots are more easily found by Horner's Method.

In the first case the roots may be calculated by a method involving Trigonometry. (See § 616, Chapter XXXIII.)

Exercise 95

Find the three roots of:

1. $x^3 + 6x^2 = 36.$
2. $3x^3 - 6x^2 - 2 = 0.$
3. $x^3 - 3x^2 - 6x - 4 = 0.$
4. $9x^3 - 54x^2 + 90x - 50 = 0.$
5. $x^3 + 3mx^2 = m^2(m+1)^2.$

6. In the case of the cubic, putting

$$L \equiv \alpha + \omega\beta + \omega^2\gamma, \quad M \equiv \alpha + \omega^2\beta + \omega\gamma,$$

show that
$$\begin{aligned} L^3 + M^3 &= 2\,\Sigma\alpha^3 - 3\,\Sigma\alpha^2\beta + 12\,\alpha\beta\gamma \\ &= -27\left(\frac{d}{a} - \frac{3\,bc}{a^2} + \frac{2\,b^3}{a^3}\right) \\ &= -\frac{27\,G}{a^3}; \\ LM &= \Sigma\alpha^2 - \Sigma\alpha\beta \\ &= -\frac{9\,H}{a^2}; \end{aligned}$$

and
$$L^3 - M^3 = -3\sqrt{-3}(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta).$$

7. From Example 6, and the relation

$$(L^3 - M^3)^2 \equiv (L^3 + M^3)^2 - 4\,L^3M^3,$$

show that $a^6(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 = -27\,(G^2 + 4\,H^3),$

and thence deduce the conditions of § 594.

595. The General Biquadratic. We shall write the general equation of the fourth degree in the form

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0. \qquad [1]$$

Put $$z = ax + b.$$

Then, $$x = \frac{z-b}{a}.$$

Substitute in [1] this expression for x and reduce,

$$z^4 + 6(ac - b^2)z^2 + 4(a^2d - 3abc + 2b^3)z$$
$$+ (a^3e - 4a^2bd + 6ab^2c - 3b^4) = 0. \qquad [2]$$

The fourth term may be written

$$a^2(ae - 4bd + 3c^2) - 3(ac - b^2)^2.$$

Put $$H \equiv ac - b^2,$$
$$G \equiv a^2d - 3abc + 2b^3,$$
and $$I \equiv ae - 4bd + 3c^2.$$

Then [2] is written in the form

$$z^4 + 6Hz^2 + 4Gz + a^2I - 3H^2 = 0, \qquad [3]$$

in which the z^3 term is wanting.

To solve this equation put

$$z = \sqrt{u} + \sqrt{v} + \sqrt{w}.$$

Square, $$z^2 = u + v + w + 2(\sqrt{uv} + \sqrt{uw} + \sqrt{vw}).$$

Transpose, and square again,

$$z^4 - 2(u + v + w)z^2 + (u + v + w)^2$$
$$= 4(uv + uw + vw) + 8z\sqrt{u}\sqrt{v}\sqrt{w}.$$

If this equation is identical with [3],

$$u + v + w = -3H,$$
$$uv + uw + vw = 3H^2 - \frac{a^2I}{4},$$
$$\sqrt{u}\sqrt{v}\sqrt{w} = -\frac{G}{2}.$$

Hence (§ 521), u, v, and w are the roots of the cubic

$$t^3 + 3Ht^2 + \left(3H^2 - \frac{a^2I}{4}\right)t - \frac{G^2}{4} = 0. \qquad [4]$$

This is known as *Euler's cubic.*

This equation may be written

$$(t+H)^3 - \frac{a^2I}{4}(t+H) + \frac{a^2HI - G^2 - 4H^3}{4} = 0,$$

or, putting $t + H = a^2\theta$, and clearing of fractions,

$$4a^3\theta^3 - Ia\theta + J = 0, \qquad [5]$$

where $J \equiv \frac{1}{a^3}(a^2HI - G^2 - 4H^3) \equiv ace + 2bcd - ad^2 - eb^2 - c^3.$

Equation [5] is called the **reducing cubic** of the biquadratic.

If θ_1, θ_2, θ_3 are the roots of this cubic, since $t = a^2\theta - H$, the four roots of equation [1] are given by

$$ax + b = \sqrt{a^2\theta_1 - H} + \sqrt{a^2\theta_2 - H} + \sqrt{a^2\theta_3 - H}. \qquad [6]$$

Since each radical may be either + or −, there are apparently eight values of x obtained from the following combinations of signs:

$$\begin{array}{cccc} +++ & ++- & +-+ & -++ \\ --- & --+ & -+- & +-- \end{array}$$

But $\sqrt{u}\sqrt{v}\sqrt{w} = -\frac{G}{2}$. Consequently, the number of admissible combinations is reduced to four.

Hence, if x_1, x_2, x_3, and x_4 are the roots of the equation

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0,$$

then $x_1 = \frac{1}{a}\left\{-b + \sqrt{at - H} + \sqrt{-at - 2H - \frac{G}{\sqrt{at - H}}}\right\},$

$$x_2 = \frac{1}{a}\left\{-b + \sqrt{at - H} - \sqrt{-at - 2H - \frac{G}{\sqrt{at - H}}}\right\},$$

$$x_3 = \frac{1}{a}\left\{-b - \sqrt{at - H} + \sqrt{-at - 2H + \frac{G}{\sqrt{at - H}}}\right\},$$

$$x_4 = \frac{1}{a}\left\{-b - \sqrt{at - H} - \sqrt{-at - 2H + \frac{G}{\sqrt{at - H}}}\right\},$$

where $t = -\frac{1}{2}\sqrt[3]{J + \frac{1}{3\sqrt{3}}\sqrt{27J^2 - I^3}} - \frac{1}{2}\sqrt[3]{J - \frac{1}{3\sqrt{3}}\sqrt{27J^2 - I^3}},$

$$H = ac - b^2,$$
$$I = ae - 4bd + 3c^2,$$
$$J = ace + 2bcd - ad^2 - eb^2 - c^3,$$

and $$G = a^2d - 3abc + 2b^3.$$

The above solution is known as *Euler's*.

In determinant form

$$H \equiv \begin{vmatrix} a & b \\ b & c \end{vmatrix}, \qquad J \equiv \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

596. Discussion of the Solution. Represent by $\alpha, \beta, \gamma, \delta$ the roots of the given biquadratic.

Then, by equation [6], we have

$$\left.\begin{aligned} a\alpha + b &= +\sqrt{u} - \sqrt{v} - \sqrt{w} \\ a\beta + b &= -\sqrt{u} + \sqrt{v} - \sqrt{w} \\ a\gamma + b &= -\sqrt{u} - \sqrt{v} + \sqrt{w} \\ a\delta + b &= +\sqrt{u} + \sqrt{v} + \sqrt{w} \end{aligned}\right\}. \qquad [7]$$

From [7], if $\theta_1, \theta_2, \theta_3$ are the roots of the reducing cubic,

$$\left.\begin{aligned} u &= a^2\theta_1 - H = \frac{a^2}{16}(\beta + \gamma - \alpha - \delta)^2 \\ v &= a^2\theta_2 - H = \frac{a^2}{16}(\gamma + \alpha - \beta - \delta)^2 \\ w &= a^2\theta_3 - H = \frac{a^2}{16}(\alpha + \beta - \gamma - \delta)^2 \end{aligned}\right\}. \qquad [8]$$

There are six cases to be considered.

I. *The four roots of the biquadratic all real and unequal.*

In this case by equations [8] u, v, w are all real. Consequently, θ_1, θ_2, θ_3 are all real, and the cubics [4] and [5] fall under the irreducible case (§ 594, I).

II. *Roots all complex and unequal.*

By § 525 the roots must be of the forms

$$h + ki,\ h - ki,\ l + mi,\ l - mi,$$

and from equations [8]

$$u = -\frac{a^2}{4}(k - m)^2,$$

$$v = -\frac{a^2}{4}(k + m)^2,$$

$$w = \frac{a^2}{4}(h - l)^2.$$

So that the roots of Euler's cubic are all real, two being negative and one positive, and the cubics [4] and [5] again fall under the irreducible case (§ 594, I).

III. *Two roots real and two complex.*

In each cubic two roots are complex and one is real.

IV. *Two roots equal, the other two unequal.*

Each of the cubics has a pair of equal roots.

V. *Two pairs of equal roots.*

Two roots of Euler's cubic vanish, the third being $-3H$. The roots of the reducing cubic are $\frac{H}{a^2}$, $\frac{H}{a^2}$, $-\frac{2H}{a^2}$.

VI. *Three roots equal.*

The roots of Euler's cubic are $-H$, $-H$, $-H$; those of the reducing cubic all vanish.

VII. *All four roots equal.*

All the roots of both cubics vanish and $H = 0$.

597. Discriminant. Comparing the reducing cubic with the cubic

$$z^3 + 3Hz + G = 0,$$

we find the discriminant of the reducing cubic to be

$$-\frac{1}{16 \times 27 a^6}(I^3 - 27J^2). \qquad (§ 594)$$

The expression $I^3 - 27J^2$ is called the **discriminant** of the biquadratic.

From the last section we obtain the following:

I. Discriminant of the reducing cubic *negative;* that is, $I^3 - 27J^2$ *positive.*

The roots of the biquadratic are either all real or all complex.

II. Discriminant of the reducing cubic *vanishes;* that is, $I^3 - 27J^2 = 0$.

The roots of the biquadratic fall under one of the following cases:

(1) Two roots equal, the other two unequal.

(2) Two pairs of equal roots. In this case $G = 0$, and

$$I = \frac{12H^2}{a^2}, \qquad J = \frac{8H^3}{a^3}.$$

(3) Three roots equal. In this case $I = 0$ and $J = 0$.

(4) Four roots equal. In this case $I = 0$, $J = 0$, $H = 0$.

III. Discriminant of the reducing cubic *positive;* that is, $I^3 - 27J^2$ *negative.*

Two of the roots of the biquadratic are real and two are complex.

598. When the left member of a biquadratic is the product of two quadratic factors with rational coefficients, the equation can be readily solved as follows:

Solve the equation

$$x^4 - 12x^3 + 12x^2 + 176x - 96 = 0.$$

Here, $a = 1$, $b = -3$; put $z = x - 3$.

Then, $$z^4 - 42z^2 + 32z + 297 = 0.$$

Compare this with

$$(z^2 + pz + q)(z^2 - pz + q') = 0,$$

and we find $$q' + q - p^2 = -42,$$

$$q' - q = \frac{32}{p},$$

$$qq' = 297.$$

Eliminating q and q', p is given by

$$p^6 - 84p^4 + 576p^2 - 1024 = 0,$$

of which two roots are found to be ± 2.

Take $p = 2$, then $q' = -11$, $q = -27$, and the equation in z is

$$(z^2 + 2z - 27)(z^2 - 2z - 11) = 0.$$

From this $$z = -1 \pm 2\sqrt{7}, \text{ or } 1 \pm 2\sqrt{3}.$$

Since $x = z + 3$, we find the four values of x to be

$$2 + 2\sqrt{7};\ 2 - 2\sqrt{7};\ 4 + 2\sqrt{3};\ 4 - 2\sqrt{3}.$$

In a similar manner, we can solve any biquadratic when the cubic in p^2 has a commensurable root.

Exercise 96

Find the four roots of:

1. $x^4 - 12x^3 + 50x^2 - 84x + 49 = 0.$
2. $x^4 - 17x^2 - 20x - 6 = 0.$
3. $x^4 - 8x^3 + 20x^2 - 16x - 21 = 0.$
4. $x^4 - 11x^3 + 46x^2 - 117x + 45 = 0.$
5. $x^4 - 7x^3 - 60x^2 + 221x - 169 = 0.$

6. Show that the biquadratic can be solved by quadratics if $G = 0$.

7. Show that the two biquadratic equations

$$ax^4 + 6cx^2 \pm 4dx + e = 0$$

have the same reducing cubic.

8. Solve the biquadratic for the two particular cases in which $I = 0$ and $J = 0$.

9. Show that if H is positive the biquadratic has either two or four complex roots.

10. Find the reducing cubic of

$$x^4 - 6ax^2 + 8x\sqrt{a^3 + b^3 + c^3 - 3abc} + (12bc - 3a^2) = 0.$$

11. Show that J vanishes for the biquadratic

$$3a(x - 2a)^4 = 2a(x - 3a)^4.$$

12. If the roots of a biquadratic are all real, and are in harmonical progression, show that the roots of Euler's cubic are in arithmetical progression.

13. Form the equation whose roots are the squares of the roots of $ax^3 + 3bx^2 + 3cx + d = 0$.

14. Form the equation whose roots are the cubes of the roots of $ax^3 + 3bx^2 + 3cx + d = 0$.

15. Form the equation whose roots are the squares of the roots of $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$.

16. Form the equation whose roots are the cubes of the roots of $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$.

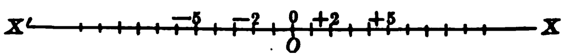
17. Show that, if $a^2I = 12H^2$ and $a^3J = 8H^3$, the biquadratic has two distinct pairs of equal roots.

CHAPTER XXXIII

COMPLEX NUMBERS

599. Representation of Scalar Numbers. Let XX' be a straight line of unlimited length. Let O be a fixed point on that line.

With any convenient unit of length measure off along the line from O to the right and to the left a series of equal distances.



Each of the points of division thus obtained represents an integer (§ 22). If the points to the right represent positive integers, those to the left represent negative integers.

The point O represents 0.

To represent a rational fraction $\frac{a}{b}$, where a and b are integers, b being positive and a either positive or negative, we divide the unit into b equal parts, and then measure off a of these parts. The point obtained lies between two of the points that represent integers.

We cannot find *exactly* the point that represents a given incommensurable number. We can, however, always find two fractions between which the given incommensurable number lies; and the point that represents the incommensurable number lies between the points that represent the two fractions.

Since the difference between the fractions can be made as small as we please, the distance between the two points that represent the fractions can be made as small as we please, and

the position of the point that represents the given incommensurable number can therefore be determined to any desired degree of accuracy.

600. The following example will illustrate the preceding argument.

The odd-numbered convergents to the periodic continued fraction

$$1+\frac{\dot{1}}{1}+\frac{\dot{1}}{2},$$

numbering from $1+\dfrac{1}{1+\frac{1}{2}}$ as the first convergent, are (§ 451)

$$\tfrac{5}{3},\ \tfrac{19}{11},\ \tfrac{71}{41},\ \tfrac{265}{153},\ \cdots, \qquad [1]$$

and the even-numbered convergents are

$$\tfrac{7}{4},\ \tfrac{26}{15},\ \tfrac{97}{56},\ \tfrac{362}{209},\ \cdots \qquad [2]$$

Let K denote the complete value of the continued fraction, and k_t denote the convergent numbered t, then (§ 449, Cor.),

$$\left.\begin{array}{l} k_{2t-3} < k_{2t-1} < K \\ k_{2t-2} > k_{2t} > K \end{array}\right\} \qquad [3]$$

and

for all positive integral values of t.

$$\therefore\ k_{2t} - k_{2t-1} > K - k_{2t-1} > 0,$$

and $$k_{2t} - k_{2t-1} > k_{2t} - K > 0.$$

Now, $$k_{2t} - k_{2t-1} < (k_2 - k_1)^t,\ \text{if}\ t > 1,$$

and $$k_2 - k_1 = \tfrac{7}{4} - \tfrac{5}{3} = \tfrac{1}{12}.$$

$$\therefore\ \frac{1}{12^t} > k_{2t} - k_{2t-1},\ \text{if}\ t > 1.$$

$$\therefore\ \frac{1}{12^t} > K - k_{2t-1} > 0,$$

and $$\frac{1}{12^t} > k_{2t} - K > 0.$$

Let M be any explicitly assigned constant number less than K, so that $K - M > 0$; then, since $K - M$ is constant and not zero, an integer m can be found such that

$$K - M > \frac{1}{12^m}.$$

Therefore, since k_{2m-1} is the convergent numbered $2m - 1$,

$$K - M > K - k_{2m-1} > 0,$$

and, therefore, $$K > k_{2m-1} > M.$$

Hence, if $K > M > \frac{5}{3}$, there can be found in series [1] a convergent which shall be greater than M but less than K, thus separating K from M, no matter how small $K - M$ may be.

Similarly, if N is an explicitly assigned number greater than K, so that $N - K > 0$, then an integer n can be found such that

$$N - K > \frac{1}{12^n}.$$

$$\therefore N - K > k_{2n} - K > 0.$$

$$\therefore N > k_{2n} > K.$$

Hence, if $\frac{7}{4} > N > K$, there can be found in series [2] a convergent which shall be less than N but greater than K, thus separating K from N, no matter how small $N - K$ may be.

Hence, there exists one number, and *only one* number, which is greater than each and every convergent in the infinite series [1] and is also less than each and every convergent in the infinite series [2], namely, the number which is the complete value of the periodic continued fraction.

Returning to the representation of numbers by points, the points that represent the convergents $\frac{5}{3}$, $\frac{19}{11}$, $\frac{71}{41}$, $\frac{265}{153}$, $\cdots$ form an endless sequence advancing from $\frac{5}{3}$, and those that represent the convergents $\frac{7}{4}$, $\frac{26}{15}$, $\frac{97}{56}$, $\frac{362}{209}$, $\cdots$ form an endless sequence retrograding from $\frac{7}{4}$. No point that lies in the first sequence

coincides with a point in the second sequence or lies between two points in it; that is, *the two sequences lie wholly without each other*, as shown by [3]. Between the first sequence and the second, *but belonging to neither of them*, there lies one point, *and only one*, namely, the point that represents the complete value of the periodic continued fraction. Every other point between $\frac{5}{3}$ and $\frac{7}{4}$ either belongs to one or other of the sequences, or lies between two points of one of them. Therefore, the point that represents K, the complete value of the periodic continued fraction, is completely determined by the sequences as their *sole* point of section.

It is now easy to determine the number K. Since the point K lies between the sequence

$$\left\{\frac{5}{3},\ \frac{19}{11},\ \frac{71}{41},\ \frac{265}{153},\ \ldots,\ \frac{a_r}{b_r},\ \ldots\right\}$$

and the sequence

$$\left\{\frac{7}{4},\ \frac{26}{15},\ \frac{97}{56},\ \frac{362}{209},\ \ldots,\ \frac{c_r}{d_r},\ \ldots\right\},$$

the point K^2 lies between the sequence

$$\left\{\left(\frac{5}{3}\right)^2,\ \left(\frac{19}{11}\right)^2,\ \left(\frac{71}{41}\right)^2,\ \left(\frac{265}{153}\right)^2,\ \ldots,\ \left(\frac{a_r}{b_r}\right)^2,\ \ldots\right\}$$

and the sequence

$$\left\{\left(\frac{7}{4}\right)^2,\ \left(\frac{26}{15}\right)^2,\ \left(\frac{97}{56}\right)^2,\ \left(\frac{362}{209}\right)^2,\ \ldots,\ \left(\frac{c_r}{d_r}\right)^2,\ \ldots\right\}.$$

The first sequence may be written

$$\left\{\left(3-\frac{2}{3^2}\right),\ \left(3-\frac{2}{11^2}\right),\ \left(3-\frac{2}{41^2}\right),\ \ldots,\ \left(3-\frac{2}{b_r^2}\right),\ \ldots\right\}. \quad [4]$$

The second sequence may be written

$$\left\{\left(3+\frac{1}{4^2}\right),\ \left(3+\frac{1}{15^2}\right),\ \left(3+\frac{1}{56^2}\right),\ \ldots,\ \left(3+\frac{1}{d_r^2}\right),\ \ldots\right\}. \quad [5]$$

(§§ 455 and 459)

Now, $$b_{r+1} = 4b_r - b_{r-1} > 3b_r.$$

$$\therefore\ b_{r+1} > 3^r b_1, \text{ and } b_1 = 3.$$

$$\therefore\ b_{r+1} > 3^{r+1}.$$

$$\therefore \frac{2}{b_{r+1}^2} < \frac{2}{9^{r+1}} < \frac{1}{4 \times 9^r}.$$

Also, $$d_{r+1} = 4d_r - d_{r-1} > 3d_r.$$

$$\therefore\ d_{r+1} > 3^r d_1, \text{ and } d_1 = 4.$$

$$\therefore\ d_{r+1} > 4 \times 3^r.$$

$$\therefore \frac{1}{d_{r+1}^2} < \frac{1}{16 \times 9^r}.$$

Let M be an explicitly assigned constant number less than 3; then, however small $3 - M$ may be, since it is greater than 0 and is constant, an integer m can be found such that

$$\frac{1}{4 \times 9^m} < 3 - M,$$

and hence a point in the sequence [4] can be found that lies *between* the points that represent M and 3.

Similarly, if N is an explicitly assigned constant number greater than 3, then, however small $N - 3$ may be, there can be found in the sequence [5] a point that lies *between* the points that represent 3 and N.

Hence, the point that represents 3 lies between the sequence [4] and the sequence [5], and *no other point lies between them;* that is, the point that represents 3 is their *sole* point of section. But the point that represents K^2 lies between the sequence [4] and the sequence [5]. Hence, the point that represents K^2 must be the point that represents 3, and therefore $K^2 = 3$.

$$\therefore K = \sqrt{3}.$$

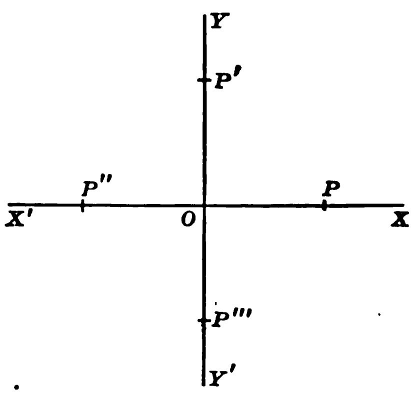
It appears, then, that *all scalar numbers may be represented by points in the line* XX′.

Conversely, ***every point in the line XX′ represents some scalar number which may be integral or fractional, commensurable or incommensurable, positive or negative.***

601. The preceding method of representing numbers assumes that the ordinal numbers, not the cardinal, are fundamental, so that the phrase the point that represents 3 is short for the phrase the point which is 3d in an endless sequence of points numbered 1st, 2d, 3d, $\cdots$; and the phrase the point that represents $\frac{5}{6}$ is short for the phrase the point which is 5th in a finite sequence of points numbered $\{$1st, 2d, 3d, 4th, 5th, 6th$\}$, say the sequence S_1^{vi}, which is itself the first sequence element in the endless sequence of sequences $\{S_1^{\text{vi}}, S_2^{\text{vi}}, S_3^{\text{vi}}, \cdots\}$.

602. Representation of Orthotomic and Complex Numbers. An orthotomic number (§ 206) cannot be represented by a point on the line XX' (§ 599), since all points on that line represent scalar numbers. We must therefore seek elsewhere for its representative point.

Let the straight lines XX' and YY' intersect at right angles at O, and mark off OP, OP', OP'', and OP''', all of the same length as in the accompanying diagram. A rotation counter-clockwise through a right angle would convert OP into OP', OP' into OP'', OP'' into OP''', and OP''' into OP, so that we may say that, taking account of *direction* as well as *length*,



$$\frac{OP'}{OP} = \frac{OP''}{OP'} = \frac{OP'''}{OP''} = \frac{OP}{OP'''}.$$

Let i denote this common ratio.

Then, $$\frac{OP'}{OP} = i,$$

and $$\frac{OP''}{OP} = \frac{OP''}{OP'} \cdot \frac{OP'}{OP} = i^2.$$

But $$\frac{OP''}{OP} = -1.$$

$$\therefore i^2 = -1.$$

Also, $$\frac{OP'''}{OP} = \frac{OP'''}{OP''} \cdot \frac{OP''}{OP'} \cdot \frac{OP'}{OP} = i^3,$$

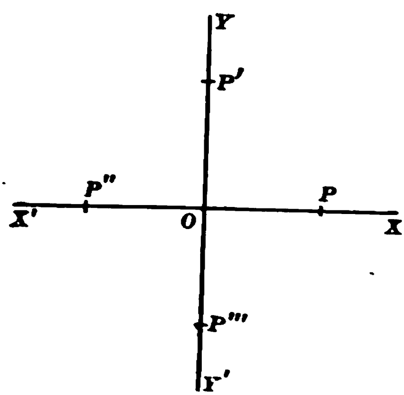
and $$\frac{OP'''}{OP} = \frac{OP''}{OP} \cdot \frac{OP'''}{OP''} = -i.$$

$$\therefore i^3 = -i.$$

Finally, $$\frac{OP}{OP} = \frac{OP}{OP'''} \cdot \frac{OP'''}{OP''} \cdot \frac{OP''}{OP'} \cdot \frac{OP'}{OP} = i^4.$$

$$\therefore i^4 = +1.$$

Hence, if we take account of direction as well as length, we have



$$OP' = i \cdot OP$$
$$= (\sqrt{-1})\,OP,$$
$$OP'' = i^2 \cdot OP$$
$$= (-1)\,OP,$$
and $$OP''' = i^3 \cdot OP$$
$$= (-\sqrt{-1})\,OP.$$

Hence, if the point P represents a scalar number a, the point P' represents the orthotomic number $a\sqrt{-1}$, and the point P''' represents the negative orthotomic number $-a\sqrt{-1}$. Thus, exactly as all scalar numbers may be represented by

points on the axis XX', so all orthotomic numbers may be represented by points on the axis YY', which cuts the axis XX' at right angles, or *orthotomically*.

Therefore, XX' is called the **axis of scalars**, and YY' is called the **axis of orthotomics**. The point O is called the **origin**.

The only point on both axes is O. This agrees with the fact that zero is the only number that may be considered either scalar or orthotomic.

Again, a and ai are measured on different lines. This agrees with the fact that a and ai are *different in kind.*

To determine a point that represents the complex number $a + b\sqrt{-1}$, determine on the scalar axis a point A that represents a, and on the orthotomic axis determine the point B that represents $b\sqrt{-1}$. Through the points A and B draw straight lines parallel to the axes. These lines intersect in a point P which represents the number $a + b\sqrt{-1}$ in the scale in which A represents a.

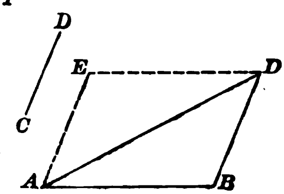
603. Vectors. When a straight line is given a definite direction and a definite length it is called a **vector**. Thus, the lines used to represent scalar numbers and those used to represent orthotomic numbers are all vectors.

Vectors need not, however, be parallel to either of the axes; they may have any direction.

The line AB, considered as a vector beginning at A and ending at B, is in general written $\overline{AB}$.

Two parallel vectors which have the same length and extend in the same direction are said to be *equal vectors.*

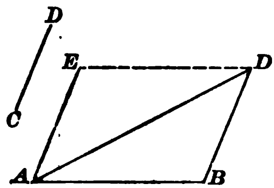
604. Vector Addition. To *add* a vector CD to a vector AB, we place C on B, keeping CD parallel to its original position, and draw AD.



$\overline{AD}$ is called the sum of the two vectors.

Then, $\overline{AD} = \overline{AB} + \overline{BD} = \overline{AB} + \overline{CD}.$

The addition here meant by the sign + is not addition of numbers, but addition of *vectors*, generally called *geometric addition.* It is evidently identical with the composition of forces.



From the dotted lines in the figure and the known properties of a parallelogram it is easily seen that

$$\overline{AD} = \overline{CD} + \overline{AB}.$$

$$\therefore \overline{AB} + \overline{CD} = \overline{CD} + \overline{AB}.$$

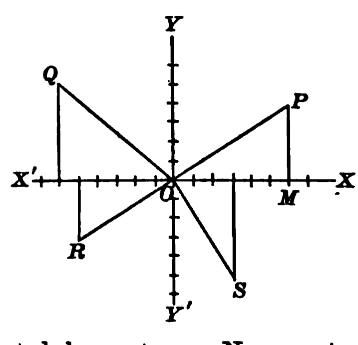
Consequently, vector addition is *commutative* (§ 36). It is easily seen that it is also *associative* (§ 36).

605. Complex Numbers. A complex number in general consists of a scalar part and an orthotomic part, and may be written (§ 212) in the *typical form* $x + yi$, where x and y are both scalar.

If we understand the sign + to indicate *geometric* addition, we shall obtain the vector that represents $x + yi$ as follows:

Lay off x on the axis of scalars from O to M. From M draw the vector $\overline{MP}$ to represent yi. Then, the vector $\overline{OP}$ is the geometric sum of the vectors $\overline{OM}$ and $\overline{MP}$, and represents the complex number $x + yi$.

Instead of the vector $\overline{OP}$ we sometimes use the point P to represent the complex number.



Thus, in the figure the vectors $\overline{OP}$, $\overline{OQ}$, $\overline{OR}$, $\overline{OS}$ or the points P, Q, R, S respectively represent the complex numbers $6 + 4i$, $-6 + 5i$, $-5 - 3i$, $3 - 5i$.

In the complex number $x + yi$, x and yi are represented by vectors. Now, vector addition is commutative. Therefore, $x + yi = yi + x$.

This is also evident from the figure.

The expression $x + yi$ is the general expression for all numbers. This expression includes zero when $x = 0$ and $y = 0$; includes all scalar numbers when $y = 0$; all orthotomic numbers when $x = 0$; all complex numbers when x and y both differ from 0.

606. Addition of Complex Numbers. Let $x + yi$ and $x' + y'i$ be two complex numbers. Their sum,

$$x + yi + x' + y'i,$$

may by the commutative law be written

$$x + x' + (y + y')i.$$

Let $\overline{OA}$ and $\overline{OB}$ be the representative vectors of $x + yi$ and $x' + y'i$. Take $\overline{AC}$ equal to $\overline{OB}$.

Then, $$\overline{OC} = \overline{OA} + \overline{OB}.$$

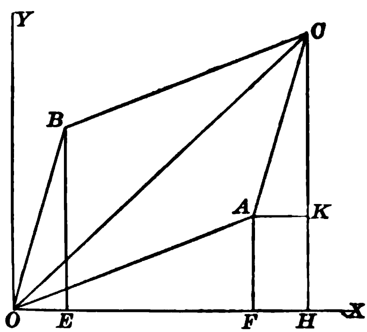
Draw the other lines in the figure.

Then, $OH = OF + FH$
$= OF + OE$
$= x + x',$

and $HC = FA + KC$
$= FA + EB$
$= yi + y'i.$

$\therefore \overline{OC} = x + x' + (y + y')i$
$= (x + yi) + (x' + y'i).$

But $\overline{OC} = \overline{OA} + \overline{OB}.$

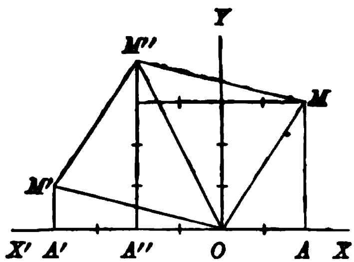


Hence, *the sum of the vectors of two complex numbers is the vector of their sum.*

Since vector addition is commutative, it follows that the addition of complex numbers is *commutative.*

The sum of two complex numbers is the geometric sum of the sum of the scalar and the sum of the orthotomic parts of the two numbers.

Find the sum of $2 + 3i$ and $-4 + i$.



$2 + 3i = \overline{OM}$, and $-4 + i = \overline{OM'}$.

If now we proceed from M, the extremity of OM, in the direction of OM' as far as the absolute value of OM', we reach the point M''.

Hence, $\overline{OM''} = -2 + 4i$, the sum of the two given complex numbers.

The same result is reached if we first find the value of $2 + (-4) = -2$. That is, if we count from O two scalar units to A'', and add to this sum $3i + i = 4i$; that is, count four orthotomic units from A'' on the perpendicular $A''M''$.

607. Modulus and Amplitude. Any complex number $x + yi$ can be written in the form

$$\sqrt{x^2 + y^2}\left(\frac{x}{\sqrt{x^2 + y^2}} + \frac{y}{\sqrt{x^2 + y^2}}i\right).$$

The expressions $\dfrac{x}{\sqrt{x^2 + y^2}}$ and $\dfrac{y}{\sqrt{x^2 + y^2}}$ may be taken as the sine and the cosine of some angle ϕ, since they satisfy the equation

$$\cos^2\phi + \sin^2\phi = 1.$$

If we put $r = \sqrt{x^2 + y^2}$, the complex number may be written

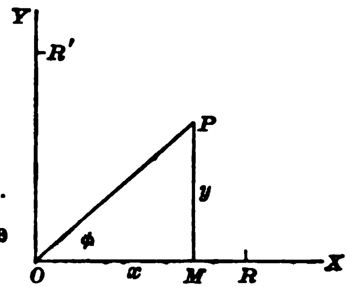
$$r(\cos\phi + i\sin\phi).$$

Since $r = \sqrt{x^2 + y^2}$, the sign of r is indeterminate. We shall, however, in this chapter take r always *positive*.

The positive number r is called the **modulus**, the angle ϕ the **amplitude**, of the complex number $x + yi$.

Let $\overline{OP}$ be the representative vector of $x + yi$. Since r is the positive value of $\sqrt{x^2 + y^2}$, it is evident that r is the *length* of OP.

On the axis OX take OR equal in length to OP and on the axis OY take OR' also equal in length to OP, then $OR = r$ and $OR' = ri$.



Also, $\cos ROP = \dfrac{OM}{OR} = \dfrac{x}{r}$,

and $\sin ROP = \dfrac{MP}{OR'} = \dfrac{yi}{ri}$.

$\therefore r(\cos ROP + i \sin ROP)$
$= x + yi = r(\cos \phi + i \sin \phi)$.

Hence, the numerical measure of the angle $ROP = \phi \pm 2n\pi$.

The above is easily seen to hold true when x and y are one or both negative.

The modulus of a scalar number is its absolute value. The amplitude of a scalar number is 0 if the number is positive, 180° if the number is negative.

The modulus of an orthotomic number ai is a. The amplitude of this number is 90° if a is positive, 270° if a is negative.

608. Since the sum of the lengths of two sides of a triangle is greater than the length of the third side, it follows, from §§ 604, 606, that, in general,

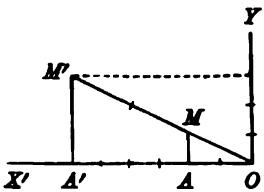
The modulus of the sum of two complex numbers is less than the sum of the moduli.

In one case, however, that in which the representative vectors are collinear, the modulus of the sum is *equal* to the sum of the moduli.

609. Multiplication of a Complex Number by a Scalar Number. Let $x + yi$ be any complex number. If the representative vector is multiplied by any scalar number a, it is easily seen from a figure that the product is $ax + ayi$.

Therefore, $$a(x + yi) = ax + ayi.$$

Hence, the multiplication of a complex number by a scalar number is *distributive.*



Multiply $-2+i$ by 3.

Take $OA=-2$ on OX', and erect at A the perpendicular $AM=1$. Then, $\overline{OM}=-2+i$. Take OM three times, and the result is $\overline{OM'}=-6+3i$, the product of $(-2+i)$ by 3.

610. Multiplication of a Complex Number by an Orthotomic Number. We have seen (§ 602) that multiplying a scalar number or an orthotomic number by i turns that number through 90°. Let us consider the effect of multiplying a complex number by i.

By the commutative, associative, and distributive laws,

$$i \times r(\cos\phi + i\sin\phi) = r(i\cos\phi - \sin\phi)$$
$$= r(-\sin\phi + i\cos\phi).$$

In Trigonometry it is shown that

$$-\sin\phi = \cos(90^\circ + \phi),$$

and
$$\cos\phi = \sin(90^\circ + \phi).$$

$$\therefore i \times r(\cos\phi + i\sin\phi) = r[\cos(90^\circ + \phi) + i\sin(90^\circ + \phi)].$$

Here, also, the effect of multiplying by i is to increase ϕ to $\phi + 90^\circ$; that is, to turn the representative vector in the positive direction through an angle of 90°.

The effect of multiplying a complex number by an orthotomic number ai is to turn the complex number through a positive angle of 90°, and also to multiply the modulus by a.

611. Multiplication of a Complex Number by a Complex Number. We come now to the general problem of the multiplication of one complex number by another. This case includes all other cases as particular cases.

Let $r(\cos\phi + i\sin\phi)$ and $r'(\cos\phi' + i\sin\phi')$ be two complex numbers.

By actual multiplication their product is

$$rr'[\cos\phi\cos\phi' - \sin\phi\sin\phi' + i(\sin\phi\cos\phi' + \cos\phi\sin\phi')].$$

In Trigonometry it is shown that

$$\cos\phi\cos\phi' - \sin\phi\sin\phi' = \cos(\phi + \phi'),$$

and

$$\sin\phi\cos\phi' + \cos\phi\sin\phi' = \sin(\phi + \phi').$$

$$\therefore r(\cos\phi + i\sin\phi) \times r'(\cos\phi' + i\sin\phi')$$
$$= rr'[\cos(\phi + \phi') + i\sin(\phi + \phi')].$$

Therefore, the *modulus* of the *product* of two complex numbers is the *product* of their moduli, and the *amplitude* of the product is the *sum* of the amplitudes.

Hence, the effect of multiplying one complex number by another is *to multiply the modulus of the first by the modulus of the second, and to turn the representative vector of the first through the amplitude of the second.*

612. Division of a Complex Number by a Complex Number.

The quotient
$$\frac{r(\cos\phi + i\sin\phi)}{r'(\cos\phi' + i\sin\phi')}$$

becomes, when both terms are multiplied by $\cos\phi' - i\sin\phi'$,

$$\frac{r[(\cos\phi\cos\phi' + \sin\phi\sin\phi') + i(\sin\phi\cos\phi' - \cos\phi\sin\phi')]}{r'(\cos^2\phi' + \sin^2\phi')}.$$

In Trigonometry it is shown that

$$\cos\phi\cos\phi' + \sin\phi\sin\phi' = \cos(\phi - \phi'),$$

$$\sin\phi\cos\phi' - \cos\phi\sin\phi' = \sin(\phi - \phi'),$$

and

$$\cos^2\phi' + \sin^2\phi' = 1.$$

$$\therefore \frac{r(\cos\phi + i\sin\phi)}{r'(\cos\phi' + i\sin\phi')} = \frac{r}{r'}[\cos(\phi - \phi') + i\sin(\phi - \phi')].$$

Hence, the *modulus* of the quotient of two complex numbers is obtained by *dividing* the modulus of the dividend by that of the divisor; and the *amplitude* of the quotient, by *subtracting* the amplitude of the divisor from that of the dividend.

613. Powers of a Complex Number. From § 611 we obtain for the case in which n is a positive integer

$$[r(\cos\phi + i\sin\phi)]^n = r^n[\cos(\phi + \phi + \cdots \text{ to } n \text{ terms}) + i\sin(\phi + \phi + \cdots \text{ to } n \text{ terms})]$$
$$= r^n(\cos n\phi + i\sin n\phi).$$

614. Roots of a Complex Number. From § 613, putting ϕ for $n\phi$, and r for r^n, we obtain

$$\left[\sqrt[n]{r}\left(\cos\frac{\phi}{n} + i\sin\frac{\phi}{n}\right)\right]^n = r(\cos\phi + i\sin\phi);$$

or
$$[r(\cos\phi + i\sin\phi)]^{\frac{1}{n}} = \sqrt[n]{r}\left(\cos\frac{\phi}{n} + i\sin\frac{\phi}{n}\right),$$

where by $\sqrt[n]{r}$ is meant the scalar positive value of the root.

The last expression gives apparently but one value for the nth root of a complex number. But we must remember that there are an unlimited number of angles which have a given sine and cosine. Thus, as shown by Trigonometry, the angles

$$\phi,\ \phi + 360°,\ \phi + 720°,\ \cdots,\ \phi + k(360°),$$

all have the same sine and the same cosine. We have, therefore, the following nth roots of $r(\cos\phi + i\sin\phi)$:

$$\sqrt[n]{r}\left(\cos\frac{\phi}{n} + i\sin\frac{\phi}{n}\right); \qquad [1]$$

$$\sqrt[n]{r}\left(\cos\frac{\phi + 360°}{n} + i\sin\frac{\phi + 360°}{n}\right); \qquad [2]$$

. . . .

$$\sqrt[n]{r}\left(\cos\frac{\phi + (n-1)360°}{n} + i\sin\frac{\phi + (n-1)360°}{n}\right); \qquad [n]$$

$$\sqrt[n]{r}\left(\cos\frac{\phi + n(360°)}{n} + i\sin\frac{\phi + n(360°)}{n}\right); \qquad [n+1]$$

. . . .

In this series the $[n+1]$th expression is the same as the *first*; the $[n+2]$th the same as the *second*; and so on.

Therefore, there are but n different nth roots, those numbered [1] to $[n]$.

From this section and the preceding section we can obtain an expression for

$$[r(\cos\phi + i\sin\phi)]^{\frac{m}{n}},$$

where $\frac{m}{n}$ is a rational fraction.

Find the twelve twelfth roots of 1.

The twelve twelfth roots of 1 are:

$$\cos 0^\circ + i \sin 0^\circ = 1; \quad [1]$$

$$\cos 30^\circ + i \sin 30^\circ = \frac{\sqrt{3}+i}{2}; \quad [2]$$

$$\cos 60^\circ + i \sin 60^\circ = \frac{1+i\sqrt{3}}{2}; \quad [3]$$

$$\cos 90^\circ + i \sin 90^\circ = i; \quad [4]$$

. . . .

. . . .

$$\cos 330^\circ + i \sin 330^\circ = \frac{\sqrt{3}-i}{2}. \quad [12]$$

615. Complex Exponents. The meaning of a complex exponent is determined by subjecting it to the same operations as a scalar exponent.

It follows that such an expression as a^{x+yi}, where a is a scalar number and $x+yi$ a complex exponent, may be simplified by resolving it into two factors, one of which is a scalar number, and the other an orthotomic power of e (§ 434).

From the ordinary rules for exponents,

$$a^{x+yi} = a^x a^{yi} = a^x (a^y)^i.$$

Put $$a^y = e^u.$$

Then, $$u = \log_e a^y = y \log_e a.$$

Now, $$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \cdots \qquad (\S\ 434)$$

Hence, $$e^{ui} = 1 + ui + \frac{u^2 i^2}{\underline{|2}} + \frac{u^3 i^3}{\underline{|3}} + \frac{u^4 i^4}{\underline{|4}} + \frac{u^5 i^5}{\underline{|5}} + \cdots$$
$$= \left(1 - \frac{u^2}{\underline{|2}} + \frac{u^4}{\underline{|4}} - \cdots\right) + i\left(u - \frac{u^3}{\underline{|3}} + \frac{u^5}{\underline{|5}} - \cdots\right).$$

By the Differential Calculus it is proved that when u is the circular measure of an angle,

$$\cos u = 1 - \frac{u^2}{\underline{|2}} + \frac{u^4}{\underline{|4}} - \frac{u^6}{\underline{|6}} + \cdots,$$
$$\sin u = u - \frac{u^3}{\underline{|3}} + \frac{u^5}{\underline{|5}} - \frac{u^7}{\underline{|7}} + \cdots,$$

each series being an infinite series.

Therefore, $$e^{ui} = \cos u + i \sin u,$$
and $$e^{x+ui} = e^x(\cos u + i \sin u).$$
Also, $$a^{x+yi} = a^x(\cos u + i \sin u)$$
$$= a^x[\cos(y \log_e a) + i \sin(y \log_e a)].$$

616. Trigonometric Solution of Cubic Equations. In the irreducible case (§ 594, I) the numerical values of the roots of a cubic equation may be found by the trigonometric tables. We have (§ 594, III)

$$ax + b = \left(\frac{-G + \sqrt{G^2 + 4H^3}}{2}\right)^{\frac{1}{3}} + \left(\frac{-G - \sqrt{G^2 + 4H^3}}{2}\right)^{\frac{1}{3}}.$$

In the case to be considered $G^2 + 4H^3$ is negative (§ 594, I).

Put $-\dfrac{G}{2} = R\cos\phi$, and $\dfrac{\sqrt{G^2 + 4H^3}}{2} = iR\sin\phi$.

$$\therefore \cos\phi = -\frac{G}{2R}, \text{ and } \sin\phi = \frac{\sqrt{G^2 + 4H^3}}{2iR}.$$

Now, by Trigonometry, $\cos^2\phi + \sin^2\phi = 1$.

$$\therefore \frac{G^2}{4R^2} - \frac{G^2 + 4H^3}{4R^2} = 1.$$

Then, $$R^2 = (-H)^3,$$

and $$R = (-H)^{\frac{3}{2}}.$$

By § 614,

$$ax + b = (-H)^{\frac{1}{2}}[(\cos\phi + i\sin\phi)^{\frac{1}{3}} + (\cos\phi - i\sin\phi)^{\frac{1}{3}}].$$

The cube roots in the right member must be so taken that their product is 1, since in § 593 $u^{\frac{1}{3}}v^{\frac{1}{3}} = -H$.

The three values of $ax + b$ are:

$$2(-H)^{\frac{1}{2}}\cos\frac{\phi}{3};$$

$$2(-H)^{\frac{1}{2}}\cos\left(\frac{\phi}{3} + 120°\right);$$

$$2(-H)^{\frac{1}{2}}\cos\left(\frac{\phi}{3} + 240°\right).$$

ϕ is given by the relation

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = -\frac{\sqrt{-(G^2 + 4H^3)}}{G}.$$

Solve the equation $z^3 - 6z + 2 = 0$.

Here, $G = 2,\ H = -2,$ and $G^2 + 4H^3 = -28.$

$$\therefore \tan\phi = -\frac{\sqrt{28}}{2} = -\sqrt{7}. \qquad \frac{\phi}{3} = 36°\ 54'\ 6''.$$

$$\log 7 = 0.84510\,n. \qquad \frac{\phi}{3} + 120° = 156°\ 54'\ 6''.$$

$$\log\tan\phi = 0.42255\,n. \qquad \frac{\phi}{3} + 240° = 276°\ 54'\ 6''.$$

$$\phi = 110°\ 42'\ 18''.$$

Then the three values of z are found by logarithms to be

$$z = 2\sqrt{2}\cos 36°\ 54'\ 6'' = 2.2618;$$
$$z = 2\sqrt{2}\cos 156°\ 54'\ 6'' = -2.6016;$$
$$z = 2\sqrt{2}\cos 276°\ 54'\ 6'' = 0.3399.$$

Check: $$-(2.2618 - 2.6016 + 0.3399) = 0;$$
$$-[2.2618 \times (-2.6016) \times 0.3399] = 2. \qquad (\S\ 521)$$

Horner's Method is, however, to be preferred to the method of this section.

617. We have in this chapter extended the term *number* to include complex expressions of the form $a + b\sqrt{-1}$. These expressions are often called imaginary quantities, although when we are considering them without attempting to give them any arithmetical interpretation, there is nothing *imaginary* about these so-called imaginaries. The collection of symbols $3 + 4\sqrt{-1}$ is, as far as symbols go, as real as the collection $3 + 4\sqrt{2}$. It is only when we seek to obtain a result arithmetically interpretable and arrive at a complex expression that cannot be interpreted, that such expression can be called in a strict sense imaginary; but under similar circumstances a fractional number or a negative number may become imaginary, while on the other hand a complex number may represent as real a solution as a positive integer represents. The following problems illustrate these statements.

(1) Two clocks begin striking at the same moment; one of the clocks strikes 6 strokes more than the other, and the number of strokes struck by one of them is double the square of the number of strokes struck by the other. Find the number of strokes struck by each clock.

(2) The temperatures indicated by two thermometers differ by 6°, and the number of degrees in the temperature indicated by one of the thermometers is double the square of the number of degrees in the temperature indicated by the other. Find the temperature indicated by each.

(3) Two men start to walk from the same place at the same moment; at the end of an hour they are 6 miles apart, and the number of miles one of them has walked is double the square of the number of miles the other has walked. Find the number of miles each has walked.

Each of these three problems yields the equations

$$y = 2x^2 \text{ and } y - x = \pm 6.$$

$$\therefore 2x^2 - x = \pm 6.$$

$$\therefore (2x^2 - x - 6)(2x^2 - x + 6) = 0.$$

$$\therefore (x-2)(2x+3)\{(2x - \tfrac{1}{2})^2 + \tfrac{47}{4}\} = 0.$$

$$\therefore x = 2 \text{ or } -1\tfrac{1}{2} \text{ or } \tfrac{1}{4}(1 \pm \sqrt{-47}).$$

$$y = 8 \text{ or } 4\tfrac{1}{2} \text{ or } \tfrac{1}{4}(-23 \pm \sqrt{-47}).$$

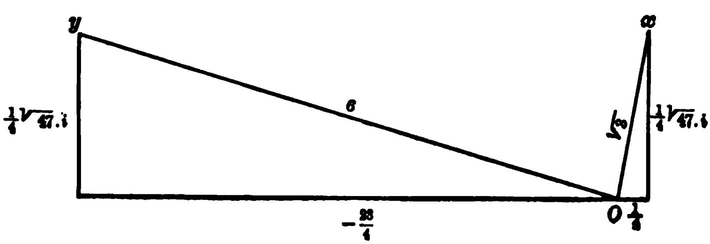
In the case of Problem (1), the only real solution is $x = 2$, $y = 8$; the other solutions are imaginary. Neither the factor $(2x + 3)$ nor the factor $\{(2x - \frac{1}{2})^2 + \frac{47}{4}\}$ can be zero in this problem

In the case of Problem (2), the real solutions are $x = 2$, $y = 8$ and $x = -1\frac{1}{2}$, $y = +4\frac{1}{2}$; the solutions $x = \frac{1}{4}(1 \pm \sqrt{-47})$, $y = \frac{1}{4}(-23 \pm \sqrt{-47})$ are both imaginary, for the factor $\{(2x - \frac{1}{2})^2 + \frac{47}{4}\}$ cannot be zero in this problem.

In the case of Problem (3), all four solutions are real. If the men walk in the same direction, the solution is $x = 2$, $y = 8$; if they walk in opposite directions, the solution is $x = -1\frac{1}{2}$, $y = 4\frac{1}{2}$; if they walk in directions obliquely transverse, the solutions are

$$x = \tfrac{1}{4}(1 \pm \sqrt{-47}), \quad y = \tfrac{1}{4}(-23 \pm \sqrt{-47}),$$

as shown in the accompanying figure, one man walking from O to x, a distance of $\sqrt{3}$ miles, while the other walks from O to y, a distance of 6 miles, the distance from x to y being then 6 miles.



Exercise 97

Find the value of:

1. $(a+bi)^4+(a-bi)^4$.

2. $\dfrac{1+i}{1+2i}+\dfrac{1-i}{1-2i}$.

3. $\dfrac{2+36i}{6+8i}+\dfrac{7-26i}{3-4i}$.

4. Show that $[(\sqrt{3}+1)+(\sqrt{3}-1)i]^3=16+16i$.

5. If $\sqrt[3]{x+yi}=a+bi$, show that

$$4(a^2-b^2)=\frac{x}{a}+\frac{y}{b}.$$

6. Find the modulus of $\dfrac{(3-4i)(2+3i)}{(6-4i)(15+8i)}$.

7. Find the three cube roots of $1+i$.

8. Find the five fifth roots of 1.

9. Find the four fourth roots of $3+4i$.

10. Solve the equation $x^3-12x+3=0$.

11. Solve the equation $2x^3-3x-1=0$.

12. Show that $(\sqrt{-1})^{\sqrt{-1}}=e^{-\frac{\pi}{4}}=0.20788$.

[redacted]